Dawson College: **Linear Algebra**: 201-105-dw-S04: Fall 2012

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Test 3

This test is graded out of 50 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Given the following vertices $A(1,1,0)$, $B(-2,2,1)$, $C(3,-3,-2)$.

a. *(3 marks)* Find the area of the triangle ΔABC using the cross-product.

b. *(3 marks)* Find the perimeter of the triangle ΔABC.

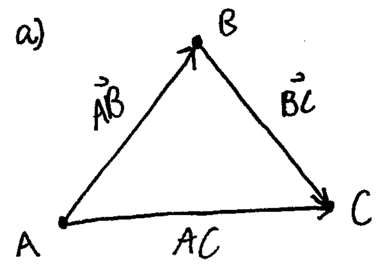
c. *(3 marks)* Find the length of the altitude from vertex B to side AC using projections.

b. *(1 mark)* Find the area of the triangle ΔABC using part c.

$\vec{AB} = B - A = (-2,2,1)-(1,1,0)$
$= (-3,1,1)$

$\vec{AC} = (3,-3,-2)-(1,1,0)$
$= (2,-4,-2)$

a)



$A = \frac{1}{2}\|\vec{AB} \times \vec{AC}\| = \frac{1}{2}\left\|\left(\begin{vmatrix} 1 & -4 \\ 1 & -2 \end{vmatrix}, -\begin{vmatrix} -3 & 2 \\ 1 & -2 \end{vmatrix}, \begin{vmatrix} -3 & 2 \\ 1 & -4 \end{vmatrix}\right)\right\|$

$\begin{matrix} -3 & 2 \\ 1 & -4 \\ 1 & -2 \end{matrix}$

$= \frac{1}{2}\|(2, -4, 10)\|$

$= \frac{1}{2}\sqrt{2^2 + (-4)^2 + 10^2} = \sqrt{4+16+100}$

$= \sqrt{\frac{120}{4}}$

$= \sqrt{30}$

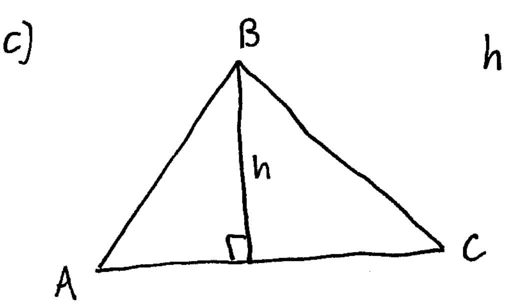
b) $\vec{BC} = C - B = (3,-3,-2) - (-2,2,1) = (5,-5,-3)$

$P = \|\vec{AB}\| + \|\vec{AC}\| + \|\vec{BC}\|$

$= \|(-3,1,1)\| + \|(2,-4,-2)\| + \|(5,-5,-3)\| = \sqrt{(-3)^2+1^2+1^2} + \sqrt{2^2+(-4)^2+(-2)^2} + \sqrt{5^2+(-5)^2+(-3)^2}$

$= \sqrt{11} + \sqrt{24} + \sqrt{59}$

c)



$h = \|\vec{w}_2\|$ where $\vec{w}_2 = \vec{AB} - \text{proj}_{\vec{AC}}\vec{AB}$

$= (-3,1,1) - \frac{(-3,1,1)\cdot(2,-4,-2)}{(2,-4,-2)\cdot(2,-4,-2)}(2,-4,-2)$

$= (-3,1,1) - \frac{-12}{24}(2,-4,-2)$

$= (-3,1,1) + \frac{1}{2}(2,-4,-2)$

$= (-3,1,1) + (1,-2,-1) = (-2,-1,0)$

so $h = \|\vec{w}_2\| = \sqrt{(-2)^2+(-1)^2+0^2} = \sqrt{5}$

d) $A = \frac{1}{2}bh = \frac{1}{2}\|\vec{AC}\|\sqrt{5} = \frac{1}{2}\sqrt{24}\sqrt{5} = \sqrt{\frac{24}{4}\cdot 5} = \sqrt{30}$

Question 2. Given

$$\begin{array}{lrcll}
\mathscr{L}_1: & (x,y,z) & = & (2+5t,1+t,-t) & t\in\mathbb{R}\\
\mathscr{L}_2: & (x,y,z) & = & (7+2t,4,10t) & t\in\mathbb{R}\\
\mathscr{L}_3: & (x,y,z) & = & (9-t,2,9-5t) & t\in\mathbb{R}\\
\mathscr{P}_1: & x-2y+3z-11 & = & 0 & \\
\mathscr{P}_2: & -5x-y+z+31 & = & 0 & \\
\mathscr{P}_3: & -3x+6y-9z+1 & = & 0 &
\end{array}$$

a. *(2 marks)* Are $\mathscr{P}_1$ and $\mathscr{L}_2$ parallel, perpendicular, or neither, justify?

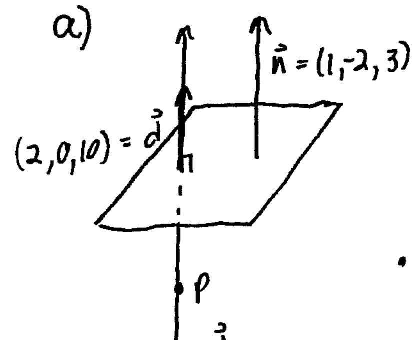
b. *(2 marks)* Are $\mathscr{P}_1$ and $\mathscr{P}_3$ parallel, perpendicular, or neither, justify?

c. *(2 marks)* Are $\mathscr{P}_2$ and $\mathscr{P}_3$ parallel, perpendicular, or neither, justify?

d. *(2 marks)* Are $\mathscr{L}_2$ and $\mathscr{L}_3$ parallel, perpendicular, or neither, justify?

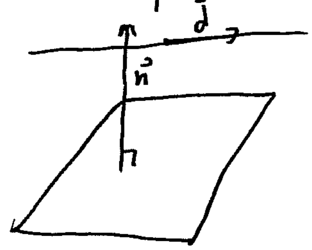
e. *(3 marks)* Find the point intersection between $\mathscr{P}_1$ and $\mathscr{L}_1$ if it exists.

a)



In order to be $\perp$ $\vec{d} = k\vec{n}$ which is not the case.

$\therefore$ not $\perp$



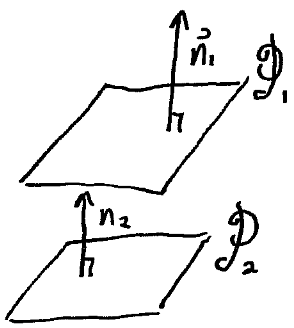
In order to be $\parallel$ $\vec{d}\cdot\vec{n} = 0$

$(2,0,10)\cdot(1,-2,3) \neq 0$

$\therefore$ not $\parallel$.

$\therefore$ neither

b)



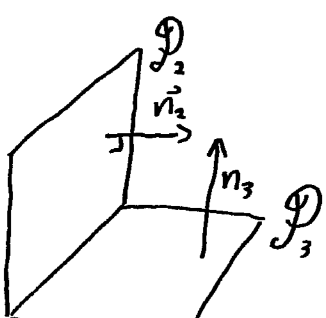
In order for $\mathcal{P}_1 \parallel \mathcal{P}_2$

$\vec{n}_1 = k\vec{n}_2$

$(1,-2,3) = \frac{-1}{3}(-3,6,-9)$

$\therefore$ $\mathcal{P}_1$ and $\mathcal{P}_2$ are parallel

c)



In order for $\mathcal{P}_2 \perp \mathcal{P}_3$

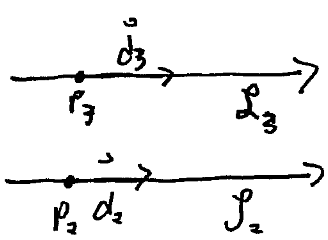
$\vec{n}_2 \cdot \vec{n}_3 = 0$

$(-5,-1,1)\cdot(-3,6,-9)$

$= 15 - 6 - 9 = 0$

$\therefore$ $\mathcal{P}_2$ and $\mathcal{P}_3$ are perpendicular.

d)



In order for $\mathcal{L}_3 \parallel \mathcal{L}_2$

$\vec{d}_3 = k\vec{d}_2$

$(-1,0,-5) = \frac{-1}{2}(2,0,10)$

$\therefore$ $\mathcal{L}_3$ and $\mathcal{L}_2$ are parallel.

e)

$$\mathcal{L}_1 \begin{cases} x = 2+5t \\ y = 1+t \\ z = -t \end{cases} \quad t \in \mathbb{R}$$

sub into $\mathcal{P}_1$

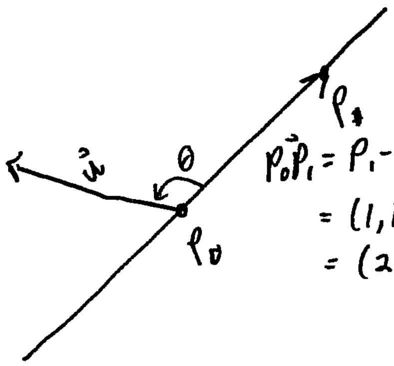
$(2+5t)-2(1+t)+3(-t)-11 = 0$

$2+5t-2-2t-3t-11 = 0$

$-11 = 0$ ↯

$\therefore$ no intersection between $\mathcal{P}_1$ and $\mathcal{L}_1$.

Question 3. *(5 marks)* Suppose that the initial point of the vector $\mathbf{u} = (1,0,-3)$ lies on the line $(x,y,z) = (-1+2t, t, 9-3t)$ where $t \in \mathbb{R}$. Find the angle in radians between the vector and the line.



$\vec{P_0P_1} = P_1 - P_0$
$= (1,1,6) - (-1,0,9)$
$= (2,1,-3)$

$\ell: \begin{cases} x = -1+2t \\ y = t \\ z = 9-3t \end{cases}$

Let $t = 0$
$(x,y,z) = (-1,0,9) = P_0$

Let $t = 1$
$(x,y,z) = (1,1,6) = P_1$

$\vec{u} \cdot \vec{P_0P_1} = \|\vec{u}\| \|\vec{P_0P_1}\| \cos\theta$

$\cos\theta = \dfrac{\vec{u} \cdot \vec{P_0P_1}}{\|\vec{u}\| \|\vec{P_0P_1}\|}$

$\cos\theta = \dfrac{(1,0,-3) \cdot (2,1,-3)}{\|(1,0,-3)\| \|(2,1,-3)\|}$

$\cos\theta = \dfrac{11}{\sqrt{10}\sqrt{14}}$

$\theta = \cos^{-1}\left(\dfrac{11}{\sqrt{140}}\right) = 22^\circ$ or 158°

$\theta = 22^\circ \left(\dfrac{\pi}{180^\circ}\right) = 0.12$ rad

or

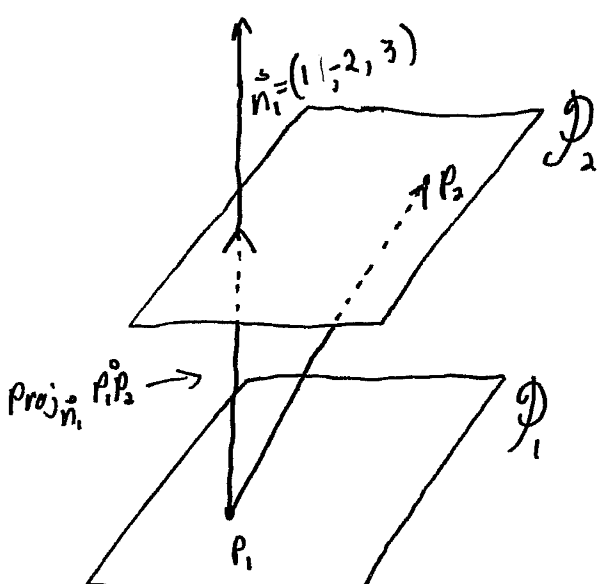
$\theta = 158^\circ \left(\dfrac{\pi}{180^\circ}\right) = 2.76$ rad

Question 4. *(5 marks)* Find the distance between the following two parallel planes

$$\begin{aligned} \mathscr{P}_1: &\quad x - 2y + 3z - 1 &= 0 \\ \mathscr{P}_2: &\quad -x + 2y - 3z + 3 &= 0. \end{aligned}$$

For a point on $\mathscr{P}_1$, let $y=0$, $z=0$, so $x - 2(0) + 3(0) - 1 = 0$, $x = 1$ $\therefore P_1(1,0,0)$

For a point on $\mathscr{P}_2$, let $y=0$, $z=0$ so $-x + 2(0) - 3(0) + 3 = 0$, $x = 3$ $\therefore P_2(3,0,0)$



So $\vec{P_1P_2} = P_2 - P_1 = (3,0,0) - (1,0,0)$
$= (2,0,0)$

$d = \|\mathrm{proj}_{\vec{n_1}} \vec{P_1P_2}\|$

$= \left\| \dfrac{(2,0,0)\cdot(1,-2,3)}{(1,-2,3)\cdot(1,-2,3)} (1,-2,3) \right\|$

$= \left\| \dfrac{2}{1+4+9} (1,-2,3) \right\| = \left\| \left(\frac{1}{7}, \frac{-2}{7}, \frac{3}{7}\right) \right\|$

$= \sqrt{\left(\frac{1}{7}\right)^2 + \left(\frac{-2}{7}\right)^2 + \left(\frac{3}{7}\right)^2}$

$= \sqrt{\dfrac{14}{7^2}} = \sqrt{\dfrac{2}{7}}$

Question 5. Given

$$\begin{aligned} \mathscr{P}_1: \quad x-3y+3z-1 &= 0 \\ \mathscr{P}_2: \quad -2x+2y-3z+3 &= 0. \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} x-3y+3z &= 1 \\ -2x+2y-3z &= -3 \end{aligned}$$

a. *(5 marks)* Find the intersection of $\mathscr{P}_1$ and $\mathscr{P}_2$.

b. *(3 marks)* Find the equation of the plane perpendicular to the intersection of $\mathscr{P}_1$ and $\mathscr{P}_2$ and that passes through $P(1,2,-3)$.

a)

$$\begin{bmatrix} 1 & -3 & 3 & 1 \\ -2 & 2 & -3 & -3 \end{bmatrix} \sim_{2R_1+R_2 \to R_2} \begin{bmatrix} 1 & -3 & 3 & 1 \\ 0 & -4 & 3 & -1 \end{bmatrix}$$

$$\sim_{4R_1 \to R_1} \begin{bmatrix} 4 & -12 & 12 & 4 \\ 0 & -4 & 3 & -1 \end{bmatrix}$$

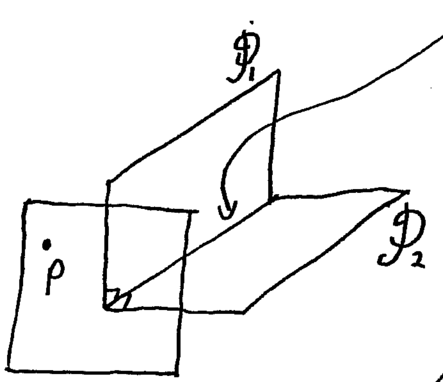
$$\sim_{-3R_2+R_1 \to R_1} \begin{bmatrix} 4 & 0 & 3 & 7 \\ 0 & -4 & 3 & -1 \end{bmatrix}$$

Let $z=t$ sub into $\begin{aligned} 4x + 3z &= 7 \\ -4y + 3z &= -1 \end{aligned}$

we get
$$x = \frac{7}{4} - \frac{3}{4}t$$
$$y = \frac{1}{4} + \frac{3}{4}t$$
$$z = t$$

$\therefore \quad (x,y,z) = \left(\frac{7}{4}, \frac{1}{4}, 0\right) + t\left(-\frac{3}{4}, \frac{3}{4}, 1\right)$

b)



$\therefore$ the normal to the new plane is a multiple of the direction vector $\vec{d} = \left(-\frac{3}{4}, \frac{3}{4}, 1\right)$ of the intersection of $\mathscr{P}_1$ and $\mathscr{P}_2$

i.e. $\vec{n} = (-3, 3, 4)$

So $\quad -3x + 3y + 4z + d = 0$

sub P to solve for d.

$$-3(1) + 3(2) + 4(-3) + d = 0$$
$$d = 9$$

$\therefore \quad -3x + 3y + 4z + 9 = 0$

Question 6. *(5 marks)* Find the point of intersection of $(x,y,z) = (1+2t, -2-3t, 3+5t)$ where $t \in \mathbb{R}$ and $(x,y,z) = (2-s, 1+2s, 19-s)$ where $s \in \mathbb{R}$.

$$l_1: \begin{cases} x = 1+2t \\ y = -2-3t \\ z = 3+5t \end{cases}$$

$$l_2: \begin{cases} x = 2-s \\ y = 1+2s \\ z = 19-s \end{cases}$$

Let $l_1 = l_2$

① $1+2t = 2-s$

② $-2-3t = 1+2s$

③ $3+5t = 19-s$

$2\times$① $+$②

$\cancel{2}+4t \cancel{-2} - 3t = 4 \cancel{-2s} + 1 \cancel{+2s}$

$t = 5$

sub into ① $\quad 1+2(5) = 2-s$

$s = 2-1-2(5) = -9$

sub s,t into ③ to verify consistency

$3+5(5) \overset{?}{=} 19-(-9)$

$28 = 28$ ✓

$\therefore$ intersection at $\quad x = 1+2(5) = 11$

$y = -2-3(5) = -17$

$z = 3+5(5) = 28$

$\therefore (11, -17, 28)$.

Question 7.

a. *(2 marks)* If $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 2$ then find a scalar triple product of **u**, **v** and **w** which is equal to -2. Justify.

b. *(2 marks)* Find a vector of length 3 which is oppositely directed to $\mathbf{w} = (1,-3,2)$.

c. *(2 marks)* For which values of m and n the vectors $\mathbf{u} = (15, m, 1)$ and $\mathbf{v} = (18, 12, n)$ are parallel.

a) $\vec{u}\cdot(\vec{w}\times\vec{v}) = -2$ since changing the order of the crossproduct changes vector by a negative.

b) $\dfrac{-3}{\|\vec{w}\|}\vec{w} = \dfrac{-3}{\sqrt{1^2+(-3)^2+2^2}}(1,-3,2) = \dfrac{-3}{\sqrt{14}}(1,-3,2) = \left(\dfrac{-3}{\sqrt{14}}, \dfrac{9}{\sqrt{14}}, \dfrac{-6}{\sqrt{14}}\right)$

c) $\vec{u} = k\vec{v}$

$(15, m, 1) = k(18, 12, n)$

$(15, m, 1) = (18k, 12k, nk)$

so

$15 = 18k \iff k = \frac{15}{18}$ ①

$m = 12k$ ②

$1 = nk$ ③

sub ① into ② & ③

$m = 12\left(\frac{15}{18}\right) = 10$

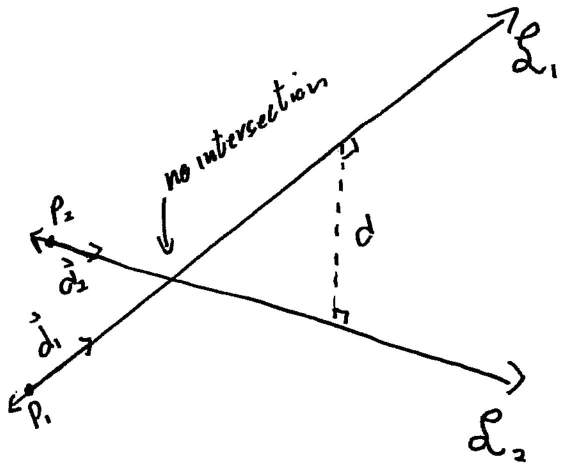
$1 = n\left(\frac{15}{18}\right)$

$n = \frac{18}{15}$.

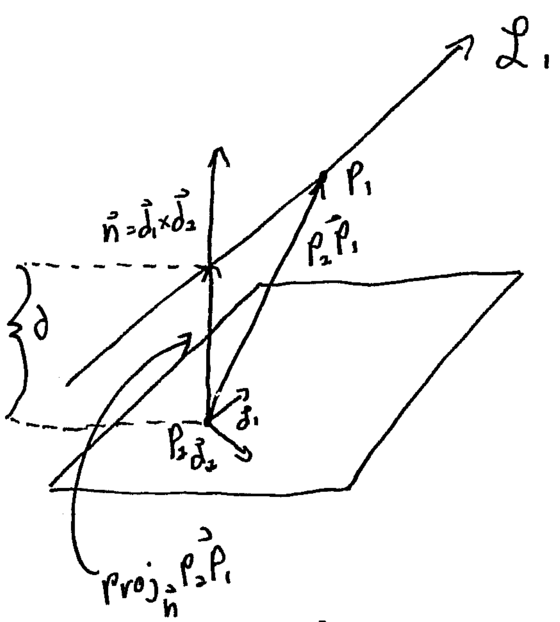
Bonus. *(5 marks)* Find the distance between

$$\begin{aligned}\mathscr{L}_1: \quad (x,y,z) &= (2+5t, 1+t, -t) \quad t \in \mathbb{R}\\ \mathscr{L}_2: \quad (x,y,z) &= (7+2t, 4, 10t) \quad t \in \mathbb{R}\end{aligned}$$

$$(x,y,z) = \overbrace{(2,1,0)}^{P_1} + t\,\overbrace{(5,1,-1)}^{\vec{d}_1}$$
$$(x,y,z) = \underbrace{(7,4,0)}_{P_2} + t\,\underbrace{(2,0,10)}_{\vec{d}_2}$$



Take $\vec{d}_1$ and $\vec{d}_2$ and construct a plane that is parallel to $\mathcal{L}_1$ and contains $\mathcal{L}_2$



$\therefore\ d = \| proj_{\vec{n}} \vec{P_2P_1} \|$

So $\vec{n} = \vec{d}_1 \times \vec{d}_2 = \left(\begin{vmatrix} 1 & 0 \\ -1 & 10 \end{vmatrix}, -\begin{vmatrix} 5 & 2 \\ -1 & 10 \end{vmatrix}, \begin{vmatrix} 5 & 2 \\ 1 & 0 \end{vmatrix} \right)$

$$\begin{array}{cc} 5 & 2 \\ 1 & 0 \\ -1 & 10 \end{array} = (10, -52, -2)$$

$$\begin{aligned}\vec{P_2P_1} &= P_1 - P_2\\ &= (2,1,0) - (7,4,0)\\ &= (-5,-3,0)\end{aligned}$$

$$\begin{aligned}d &= \| proj_{\vec{n}} \vec{P_2P_1} \|\\
&= \left\| \frac{(-5,-3,0)\cdot(10,-52,-2)}{(10,-52,-2)\cdot(10,-52,-2)}(10,-52,-2) \right\|\\
&= \left\| \frac{106}{2808}(10,-52,-2) \right\|\\
&= \left\| \left(\frac{265}{702}, \frac{-53}{27}, \frac{-53}{702} \right) \right\|\\
&= \sqrt{\left(\frac{265}{702}\right)^2 + \left(\frac{-1378}{702}\right)^2 + \left(\frac{-53}{702}\right)^2}\\
&= \sqrt{\frac{1971918}{(702)^2}}\\
&= \sqrt{\frac{2809}{702}}\end{aligned}$$