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Test 1

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Formulae:

$\sum_{i=1}^{n} c = cn$ where c is a constant $\quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Question 1. *(5 marks)* Evaluate the definite integral of $f(x) = -x^2 + x + 1$ on $[-1, 2]$ using the definition of the definite integral.

$$\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{n} = \frac{3}{n}$$

$$x_i = a + i\Delta x = -1 + \frac{3i}{n}$$

$$\int_{-1}^{2} f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-1 + \frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[-\left(-1 + \frac{3i}{n}\right)^2 + \left(-1 + \frac{3i}{n}\right) + 1\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[-\left(1 - \frac{6i}{n} + \frac{9i^2}{n^2}\right) - 1 + \frac{3i}{n} + 1\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[-1 + \frac{9i}{n} - \frac{9i^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \left[\sum_{i=1}^{n} -1 + \sum_{i=1}^{n} \frac{9i}{n} - \sum_{i=1}^{n} \frac{9i^2}{n^2}\right]$$

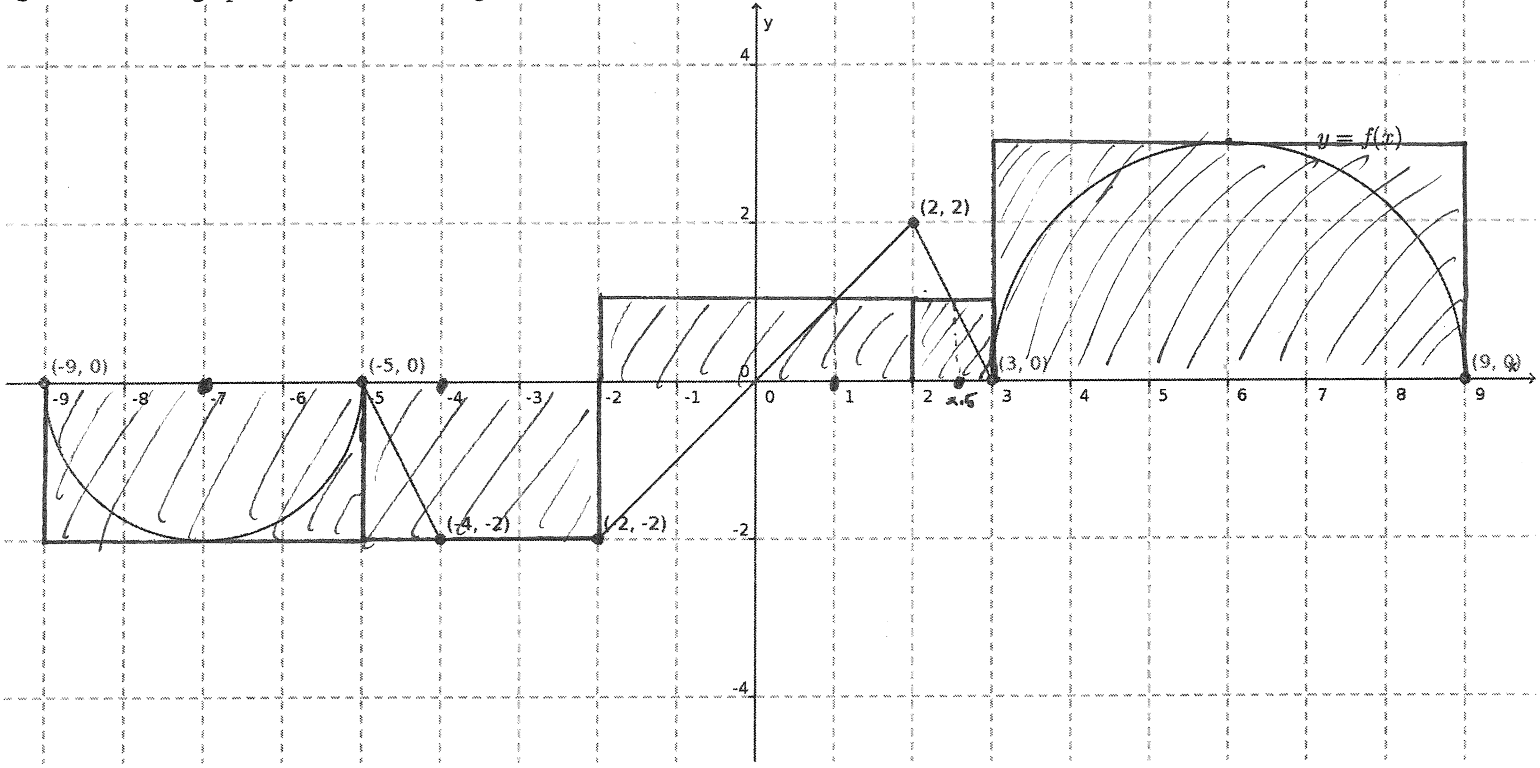
$$= \lim_{n\to\infty} \frac{3}{n} \left[-n + \frac{9}{n}\sum_{i=1}^{n} i - \frac{9}{n^2}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \left[-n + \frac{9}{\cancel{n}}\frac{\cancel{n}(n+1)}{2} - \frac{\cancel{9}^{3}}{n^{\cancel{2}}}\frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}\,2}\right]$$

$$= \lim_{n\to\infty} \left[\frac{-3\cancel{n}}{\cancel{n}} + \frac{27(n+1)}{2n} - \frac{9(n+1)}{2n}\frac{(2n+1)}{n}\right]$$

$$= -3 + \frac{27}{2} \quad \frac{9}{\cancel{2}} \cdot \frac{\cancel{2}}{1} = -3 + \frac{27}{2} - 9 = \frac{3}{2}$$

Question 2. The graph of f consists of straight lines and two semicircles.



a. *(3 marks)* Find the Riemann sum for $f(x)$, if the partition points are -9, -5, -2, 2, 3, 9 and the sample points are -7, -4, 1, 2.5, 6.

b. *(1 mark)* Draw the approximating rectangles.

a) $\sum_{i=1}^{5} f(x_i)\Delta x_i = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + f(x_3)\Delta x_3 + f(x_4)\Delta x_4 + f(x_5)\Delta x_5$

$= (-2)(4) + (-2)(3) + (1)(4) + 1(1) + 3(6)$

$= -8 - 6 + 4 + 1 + 18$

$= 9$

Question 3. *(5 marks)* If

$$\int_2^3 (f(x) - 1)\, dx = 2, \quad \int_2^1 3f(x)\, dx = 4,$$

and $f(-x) = -f(x)$ for all $x \in \mathbb{R}$ find

$$\int_{-1}^3 f(x)\, dx. = \underbrace{\int_{-1}^1 f(x)dx}_{=0} + \int_1^3 f(x)dx$$

$= \int_1^3 f(x)dx$ since $f(x)$ is odd

$= \int_1^2 f(x)dx + \int_2^3 f(x)dx$

$= 3 - \frac{4}{3} = \frac{5}{3}$

$\int_2^3 f(x)dx - \int_2^3 1\,dx = 2$

$\int_2^3 f(x)dx - [x]_2^3 = 2$

$\int_2^3 f(x)dx - [3-2] = 2$

$\int_2^3 f(x)dx = 3$

$3\int_2^1 f(x)dx = 4$

$\int_2^1 f(x)dx = 4/3$

$-\int_1^2 f(x)dx = 4/3$

$\int_1^2 f(x)dx = -4/3$

Question 4. *(5 marks)* Evaluate the definite integral:

$$\int_0^{1/2} \frac{\arcsin x}{\sqrt{1-x^2}}\, dx = \int_0^{\pi/6} u\, du = \left[\frac{u^2}{2}\right]_0^{\pi/6}$$

$u = \arcsin x$

$du = \frac{1}{\sqrt{1-x^2}} dx$

$$= \frac{(\pi/6)^2}{2} - \frac{0^2}{2}$$

$u(1/2) = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$$= \frac{\pi^2}{72}$$

$u(0) = \arcsin(0) = 0$

Question 5. Given

$$h(x) = \int_{\arcsin \pi x}^{\sin e^x} (\arcsin u) \sin u\, du = \int_{\arcsin \pi x}^{c} (\arcsin u) \sin u\, du + \int_c^{\sin e^x} (\arcsin u) \sin u\, du$$

a. *(2 marks)* Rewrite $h(x)$ as the sum of two integrals with a constant as the lower bound.

b. *(1 mark)* Rewrite the two integrals of part a. as composite functions with an integral as the outer function.

c. *(2 marks)* Using part b. and the 2nd FTC determine $h'(x)$.

$$= -\int_c^{\arcsin \pi x} (\arcsin u) \sin u\, du + \int_c^{\sin e^x} (\arcsin u) \sin u\, du$$

$$= -f(g_1(x)) + f(g_2(x))$$

where $f(x) = \int_c^x (\arcsin u) \sin u\, du$ $\quad g_1(x) = \arcsin \pi x$

$g_2(x) = \sin e^x$

then

$h'(x) = -f'(g_1(x)) g_1'(x) + f'(g_2(x)) g_2'(x)$

$f'(x) = (\arcsin x) \sin x$ by 2nd FTC

$g_1'(x) = \frac{1}{\sqrt{1-(\pi x)^2}} \cdot \pi$

$g_2'(x) = (\cos e^x) e^x$

$$h'(x) = -\left((\arcsin(\arcsin \pi x)) \sin \arcsin \pi x\right) \frac{\pi}{\sqrt{1-(\pi x)^2}} + (\arcsin \sin e^x)(\sin(\sin e^x))(\cos e^x) e^x$$

Question 6. Given

$$f(x) = \sec x - 1, \quad [0, \pi/3]$$

a. *(2 marks)* Find the average value of f on the given interval.

b. *(2 marks)* Find c such that $f_{ave} = f(c)$.

c. *(1 mark)* Sketch the graph of f and a rectangle whose area is the same as the area under the graph of f.

a) $f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\pi/3 - 0}\int_0^{\pi/3} \sec x - 1\,dx = \frac{3}{\pi}\Big[\ln|\sec x + \tan x| - x\Big]_0^{\pi/3}$

$= \frac{3}{\pi}\Big[\Big[\ln|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}| - \frac{\pi}{3}\Big] - \Big[\ln|\sec 0 + \tan 0| - 0\Big]\Big]$

$= \frac{3}{\pi}\Big[\ln|2 + \sqrt{3}| - \frac{\pi}{3}\Big]$

$= \frac{3}{\pi}\ln|2+\sqrt{3}| - 1$

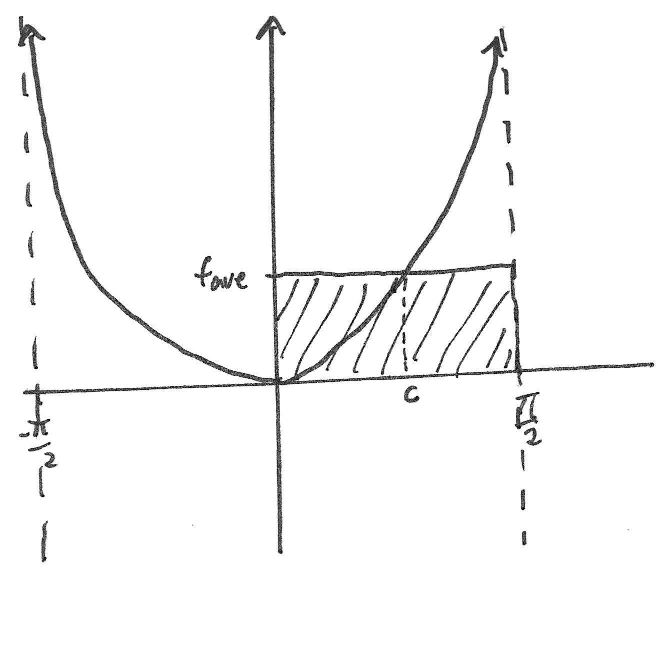
b) $f_{ave} = f(c)$

$\frac{3}{\pi}\ln(2+\sqrt{3}) - 1 = \sec c - 1$

$\frac{3}{\pi}\ln(2+\sqrt{3}) = \frac{1}{\cos c}$

$\cos c = \frac{1}{\frac{3}{\pi}\ln(2+\sqrt{3})}$

$c = \arccos\left(\frac{\pi}{3\ln(2+\sqrt{3})}\right) \approx 37^{\circ}$



prove. function

Question 7. *(5 marks)* If f is an odd continuous on $[-a,a]$, then

$$\int_{-a}^{a} f(x)\,dx = 0$$

Since $f(x)$ is odd $f(-x) = -f(x)$

so $f(x) = -f(-x)$

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

$$= -\int_{0}^{-a} f(x)dx + \int_{0}^{a} f(x)dx$$

$$= -\int_{0}^{-a} -f(-x)dx + \int_{0}^{a} f(x)dx$$

$u = -x$ $\quad u(-a) = -(-a) = a$

$du = -dx$ $\quad u(0) = -0 = 0$

$$= \int_{0}^{a} f(u)(-du) + \int_{0}^{a} f(x)dx$$

$$= -\int_{0}^{a} f(u)du + \int_{0}^{a} f(x)dx = 0$$

Question 8. *(5 marks)* Evaluate the indefinite integral

$$\int \frac{e^x + e^{2x}}{1+e^{2x}}\,dx = \int \frac{(1+e^x)e^x}{1+(e^x)^2}dx = \int \frac{1+u}{1+u^2}du$$

$u = e^x$

$du = e^x dx$

$$= \int \frac{1}{1+u^2}du + \int \frac{u}{1+u^2}du$$

$v = 1+u^2$

$dv = 2u\,du$

$\frac{dv}{2} = u\,du$

$$= \arctan u + \int \frac{1}{v}\frac{dv}{2}$$

$$= \arctan e^x + \frac{1}{2}\ln|v| + c$$

$$= \arctan e^x + \frac{1}{2}\ln|1+u^2| + c$$

$$= \arctan e^x + \frac{1}{2}\ln|1+e^{2x}| + c$$

Bonus Question. *(3 marks)*
The function F is defined on $[-9,9]$ as

$$F(z) = \int_{-9}^{z} f(x)\,dx.$$

where $f(x)$ is the graph of **Question 2**. Discuss where the second derivative test is conclusive and where it fails. (Hint: use the graph of $f(x)$).

$F'(z) = f(z)$ by 2^{nd} FTC since $f(x)$ is continuous

The critical points of $F(z)$ are when $F'(z) = 0$ in this case where $f(z) = 0$.

So the critical points are at $z = -9, -5, 0, 3, 9$

The second derivative tests fails at, -9 and 9 because they are endpoints and at $-5, 3$ because $f(z)$ has cusp at those points.

The second derivative test does not fail at 0 where $F''(0) = f'(0) = 1 > 0$ $\therefore$ at $z=0$ $F(z)$ has a minimum.