Dawson College: Calculus II (SCIENCE): 201-NYB-05-S03: Fall 2014

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Test 2

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Evaluate the integral.

$$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right)\,dx$$

$u = \arctan\left(\frac{1}{x}\right)$ $\quad du = \frac{1}{1+\left(\frac{1}{x}\right)^2}\frac{-1}{x^2}\,dx$

$v = x$ $\quad dv = dx$

$$= \Big[uv\Big]_1^{\sqrt{3}} - \int_1^{\sqrt{3}} v\,du$$

$$= \left[x\arctan\left(\frac{1}{x}\right)\right]_1^{\sqrt{3}} - \int_1^{\sqrt{3}} x\,\frac{1}{1+\left(\frac{1}{x}\right)^2}\,\frac{-1}{x^2}\,dx$$

$$= \sqrt{3}\arctan\left(\frac{1}{\sqrt{3}}\right) - 1\arctan\left(\frac{1}{1}\right) + \int_1^{\sqrt{3}} \frac{x}{x^2+1}\,dx$$

$$= \sqrt{3}\,\frac{\pi}{6} - \frac{\pi}{4} + \left[\frac{1}{2}\ln|x^2+1|\right]_1^{\sqrt{3}}$$

$$= \frac{\sqrt{3}\pi}{6} - \frac{\pi}{4} + \frac{1}{2}\ln 4 - \frac{1}{2}\ln 2$$

Question 2. *(5 marks)* Evaluate the integral.

$$\int_0^{\pi/6} \sqrt{1+\cos 2x}\, dx$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \int_0^{\pi/6} \sqrt{1+2\cos^2 x - 1}\, dx$$

$$2\cos^2 x = 1 + \cos 2x$$

$$\cos 2x = 2\cos^2 x - 1$$

$$= \int_0^{\pi/6} \sqrt{2\cos^2 x}\, dx$$

$$= \int_0^{\pi/6} \sqrt{2}\, |\cos x|\, dx$$

$$= \sqrt{2} \int_0^{\pi/6} \cos x\, dx$$ since $\cos x$ is positive on $[0, \pi/6]$

$$= \sqrt{2} \left[\sin x\right]_0^{\pi/6}$$

$$= \sqrt{2} \left[\sin \frac{\pi}{6} - \sin 0\right]$$

$$= \sqrt{2}\ \frac{1}{2}$$

$$= \frac{\sqrt{2}}{2}$$

Question 3. *(5 marks)* Evaluate the integral.

$$\int \sqrt{5+4x-x^2}\,dx$$

$$= \int \sqrt{9-(x-2)^2}\,dx$$

$$= \int \sqrt{9-(3\sin\theta)^2}\;3\cos\theta\,d\theta$$

$$= \int \sqrt{9-9\sin^2\theta}\;3\cos\theta\,d\theta$$

$$= \int \sqrt{9(1-\sin^2\theta)}\;3\cos\theta\,d\theta$$

$$= 3\int \sqrt{9\cos^2\theta}\;\cos\theta\,d\theta$$

$$= 3\int 3\,|\cos\theta|\,\cos\theta\,d\theta$$

$$= 9\int \cos\theta\,\cos\theta\,d\theta \quad \text{discussed in class}$$

$$= 9\int \cos^2\theta\,d\theta$$

$$= 9\int \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{9}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

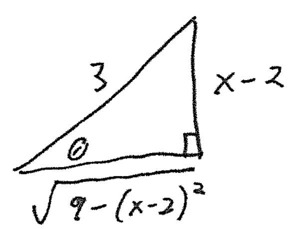
$$= \frac{9}{2}\left[\arcsin\left(\frac{x-2}{3}\right) + \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right] + C$$

$$= \frac{9}{2}\left[\arcsin\left(\frac{x-2}{3}\right) + \frac{x-2}{3}\,\frac{\sqrt{9-(x-2)^2}}{3}\right] + C$$

$$5+4x-x^2 = -x^2+4x+5$$
$$= -\left[x^2-4x-5\right]$$
$$= -\left[x^2-4x+4-4-5\right]$$
$$= -\left[(x-2)^2-9\right]$$
$$= 9-(x-2)^2$$

$$x-2 = 3\sin\theta$$
$$dx = 3\cos\theta\,d\theta$$

$$\frac{x-2}{3} = \sin\theta$$
$$\theta = \arcsin\left(\frac{x-2}{3}\right)$$



$$\cos\theta = \frac{\sqrt{9-(x-2)^2}}{3}$$

Question 4. *(5 marks)* Evaluate the integral.

$$\int \frac{x^2+1}{(x-3)(x-2)^2}\,dx$$

$$\frac{x^2+1}{(x-3)(x-2)^2} = \frac{A_1}{x-3} + \frac{A_2}{x-2} + \frac{A_3}{(x-2)^2}$$

$$x^2+1 = A_1(x-2)^2 + A_2(x-3)(x-2) + A_3(x-3)$$

Let $x=3$ $\quad 10 = A_1(3-2)^2$

$10 = A_1$

Let $x=2$ $\quad 5 = -A_3$

$A_3 = -5$

Let $x=0$ $\quad 1 = 10(-2)^2 + A_2(-3)(-2) - 3(-5)$

$1 = 40 + 6A_2 + 15$

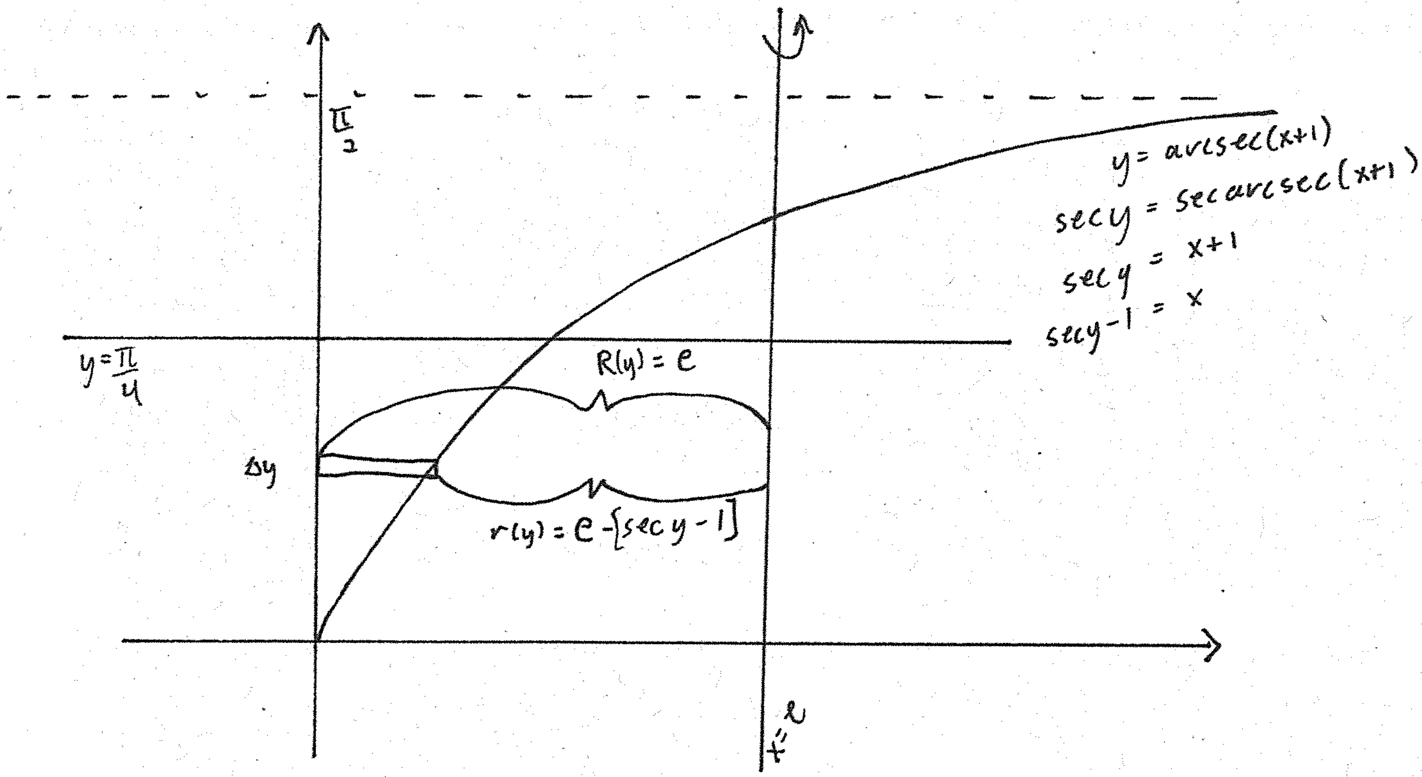
$-54 = 6A_2$

$A_2 = -9$

$$\int \frac{x^2+1}{(x-3)(x-2)^2}\,dx = \int \frac{10}{x-3} + \frac{-9}{x-2} + \frac{-5}{(x-2)^2}\,dx$$

$$= 10\ln|x-3| - 9\ln|x-2| + 5\frac{1}{x-2} + C$$

Question 5. *(5 marks)* Set up the integral to find the volume of the solid obtained from the region bounded by the graphs of $y = \operatorname{arcsec}(x+1)$, $y = \frac{\pi}{4}$, $x = 0$ rotated about the line $x = e$. Sketch the region, draw a representative rectangle, write a representative element and the integral.



$$\Delta V = \pi\left[(R(y))^2 - (r(y))^2\right]\Delta y = \pi\left[e^2 - (e - \sec y + 1)^2\right]\Delta y$$

$$V = \int_0^{\pi/4} \pi\left[e^2 - (e - \sec y + 1)^2\right] dy$$

Question 6. *(5 marks)* Determine whether the integral is convergent or divergent. Evaluate the integral if convergent.

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b \to 1^-} \int_0^b \frac{dx}{\sqrt{1-x^2}}$$

$$= \lim_{b \to 1^-} \Big[\arcsin x\Big]_0^b$$

$$= \lim_{b \to 1^-} \Big[\overset{\nearrow \pi/2}{\arcsin b} - \overset{0}{\overset{\shortparallel}{\arcsin 0}}\Big]$$

$$= \frac{\pi}{2}$$

Question 7. *(5 marks)* Use the Comparison Theorem to determine whether the integral is convergent or divergent.

$$\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$$

Let $f(x) = \dfrac{\arctan x}{2+e^x} \leq \dfrac{\pi/2}{2+e^x} \leq \dfrac{\frac{\pi}{2}}{e^x} = \frac{\pi}{2} e^{-x} = g(x)$

$$\int_0^\infty \frac{\pi}{2} e^{-x}\,dx = \lim_{b\to\infty} \int_0^b \frac{\pi}{2} e^{-x}\,dx = \lim_{b\to\infty} \left[-\frac{\pi}{2} e^{-x} \right]_0^b$$

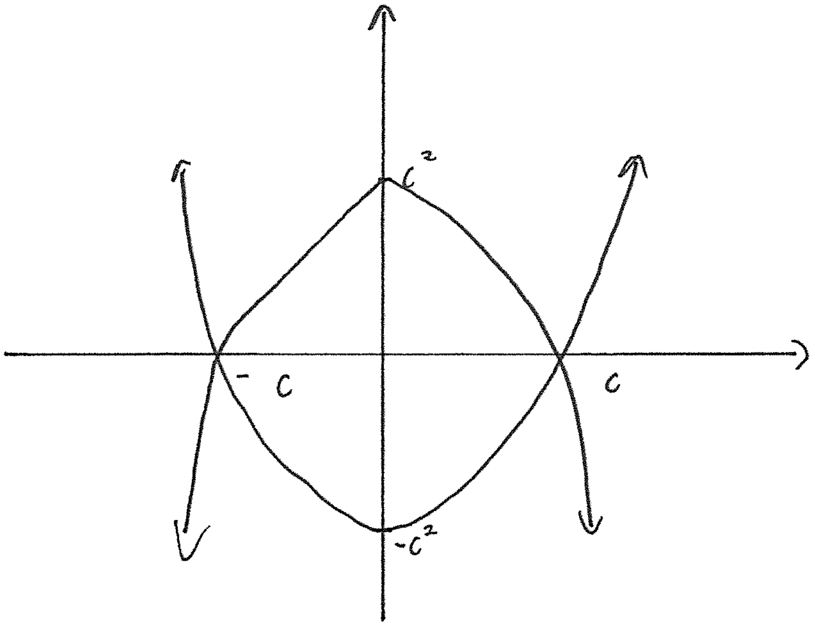
$$= \lim_{b\to\infty} \left[\cancelto{0}{-\frac{\pi}{2} e^{-b}} + \frac{\pi}{2} e^{0} \right]$$

$$= \frac{\pi}{2}$$

$\therefore$ converges

$\therefore$ by comparison theorem $\displaystyle\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$ converges.

Question 8. *(5 marks)* Find the value(s) of c such that the area of the region bounded by the parabolas $y = x^2 - c^2$ and $y = c^2 - x^2$ is 1936.



X-int:

$0 = x^2 - c^2$ $\quad$ $0 = c^2 - x^2$

$x = \pm c$ $\quad$ $x = \pm c$

$$1936 = \int_{-c}^{c} c^2 - x^2 - [x^2 - c^2]\,dx$$

$$1936 = \int_{-c}^{c} 2c^2 - 2x^2\,dx$$

$$1936 = \left[2c^2x - \frac{2x^3}{3}\right]_{-c}^{c}$$

$$1936 = \left[2c^2c - \frac{2(+c)^3}{3}\right] - \left[2c^2(-c) - \frac{2(-c)^3}{3}\right]$$

$$1936 = 2c^3 - \frac{2}{3}c^3 + 2c^3 - \frac{2}{3}c^3$$

$$1936 = \frac{8}{3}c^3$$

$$726 = c^3$$

$$c = \sqrt[3]{726}$$

Bonus Question. *(3 marks)*

Q

$$\int_e^\infty \frac{1}{z^2}\int_e^z f(x)\,dx\,dz$$

$u = \int_e^z f(x)dz \qquad du = f(z)dz$

$$= \lim_{b\to\infty}\int_e^b \frac{1}{z^2}\int_e^z f(x)\,dx\,dz \qquad v = \frac{-1}{z} \qquad dv = \frac{1}{z^2}dz$$

$$= \lim_{b\to\infty}\left[\left[\frac{-1}{z}\int_e^z f(z)dz\right]_e^b - \int_e^b -\frac{1}{z}f(z)dz\right]$$

$$= \lim_{b\to\infty}\left[\frac{-1}{b}\int_e^b f(z)dz + \underbrace{\frac{1}{e}\int_e^e f(z)dz} + \int_e^b \frac{f(z)}{z}dz\right]$$

$$= \lim_{b\to\infty}\left[\frac{-\int_e^b f(z)dz}{b} + \overbrace{\int_e^b \frac{f(z)}{z}dz}^{0}\right] = 0$$

I.F. $\frac{-\infty}{\infty}$

$$\overset{\hat{H}}{=} \lim_{b\to\infty}\frac{-f(b)}{1}$$

$= 0$

and $\lim_{x\to\infty}\int_e^x f(t)dt = \infty$

Q: Given $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\int_e^x \frac{f(t)}{t}dt = 0$ determine whether the integral is convergent or divergent. Evaluate the integral if convergent.