Dawson College: Calculus II (Science): 201-NYB-05-S03: Fall 2014

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Test 3

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=0}^{\infty}\left[\pi\left(\frac{e}{\pi}\right)^n+\left(\frac{e}{n+1}-\frac{e}{n+2}\right)\right]$$

$$\sum_{n=0}^{\infty}\pi\left(\frac{e}{\pi}\right)^n=\frac{a}{1-r} \quad \text{since } |r|=\frac{e}{\pi}<1$$

$$=\frac{\pi}{1-\frac{e}{\pi}}=\frac{\pi}{\frac{\pi-e}{\pi}}=\frac{\pi^2}{\pi-e}$$

$$\sum_{n=0}^{\infty}\left(\frac{e}{n+1}-\frac{e}{n+2}\right)$$ Lets look at the partial sum

$$S_n=a_0+a_1+a_2+\dots+a_{n-2}+a_{n-1}+a_n$$

$$=\left[\frac{e}{1}-\cancel{\frac{e}{2}}\right]+\left[\cancel{\frac{e}{2}}-\cancel{\frac{e}{3}}\right]+\left[\cancel{\frac{e}{3}}-\frac{e}{4}\right]+\dots+\left[\frac{e}{n-1}-\cancel{\frac{e}{n}}\right]+\left[\cancel{\frac{e}{n}}-\cancel{\frac{e}{n+1}}\right]+\left[\cancel{\frac{e}{n+1}}-\frac{e}{n+2}\right]$$

$$=\left[e-\frac{e}{n+2}\right]$$

$$S=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left[e-\cancelto{0}{\frac{e}{n+2}}\right]=e$$

$$\sum_{n=0}^{\infty}\left[\pi\left(\frac{e}{n}\right)^n+\left(\frac{e}{n+1}-\frac{e}{n+2}\right)\right]=\sum_{n=0}^{\infty}\pi\left(\frac{e}{\pi}\right)^n+\sum_{n=0}^{\infty}\left(\frac{e}{n+1}-\frac{e}{n+2}\right)$$

$$=\frac{\pi^2}{\pi-e}+e$$

Question 2. *(5 marks)* Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=2014}^{\infty} \sqrt[n]{2}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{2} = \lim_{n\to\infty} 2^{\frac{1}{n}} = 2^0 = 1 \neq 0$$

$\therefore$ diverges by n^{th} term divergence test.

Question 3. *(5 marks)* Determine whether the series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{1+\cos n}{2^n}$$

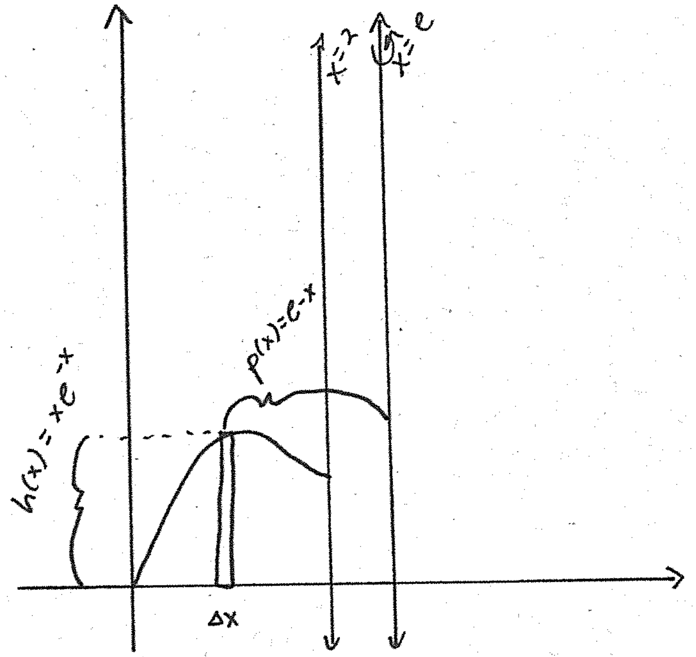
Let $a_n = \dfrac{1+\cos n}{2^n}$

$$0 \le a_n = \frac{1+\cos n}{2^n} \le \frac{1+1}{2^n} = \frac{2}{2^n} = 2\left(\frac{1}{2}\right)^n = b_n$$

where $\sum\limits_{n=0}^{\infty} b_n$ is convergent since it is a geometric series where $|r| = \left|\frac{1}{2}\right| < 1$

$\therefore \sum\limits_{n=0}^{\infty} a_n$ is convergent by the comparison test since $0 \le a_n \le b_n$ and $\sum\limits_{n=0}^{\infty} b_n$ is convergent.

Question 4. *(5 marks)* Set up the integral to find the volume of the solid obtained from the region bounded by the graphs of $y = xe^{-x}$, $y = 0$, $x = 2$ rotated about the line $x = e$. Sketch the region, draw a representative rectangle, write a representative element and the integral.



$$\Delta V = 2\pi\, p(x)\, h(x)\, \Delta x$$
$$= 2\pi\,(e-x)(xe^{-x})\,\Delta x$$

$$V = \int_0^2 2\pi\,(e-x)(xe^{-x})\,dx$$

Question 5. *(5 marks)* Find the exact length of the curve

$$y = \frac{x^3}{3} + \frac{1}{4x}, \quad 1 \leq x \leq 2$$

$$y' = \frac{3x^2}{3} - \frac{1}{4x^2} = x^2 - \frac{1}{4x^2}$$

$$S = \int_1^2 \sqrt{1 + (y')^2}\, dx$$

$$= \int_1^2 \sqrt{1 + \left(x^2 - \frac{1}{4x^2}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{1 + x^4 - \frac{1}{2} + \frac{1}{16x^4}}\, dx$$

$$= \int_1^2 \sqrt{x^4 + \frac{1}{2} + \frac{1}{16x^4}}\, dx$$

$$= \int_1^2 \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2}\, dx$$

$$= \int_1^2 \left| x^2 + \frac{1}{4x^2} \right| dx$$

$$= \int_1^2 x^2 + \frac{1}{4x^2}\, dx$$

$$= \left[\frac{x^3}{3} - \frac{1}{4x} \right]_1^2$$

$$= \left[\frac{2^3}{3} - \frac{1}{4(2)} \right] - \left[\frac{1^3}{3} - \frac{1}{4(1)} \right]$$

$$= \left[\frac{8}{3} - \frac{1}{8} \right] - \frac{1}{3} + \frac{1}{4} = \frac{59}{24}$$

Question 6.

a. *(2 marks)* Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

$$\left\{\frac{2}{1}, \frac{4}{1(2)}, \frac{8}{1(2)(3)}, \frac{16}{1(2)(3)(4)}, \frac{32}{1(2)(3)(4)(5)}, \ldots\right\}$$

b. *(3 marks)* Determine whether the sequence in part a. converges or diverges. If it converges, find the limit.

a)

$$a_n = \frac{2^n}{n!}$$

b)

$$b_n = 0 \le \underbrace{\frac{2^n}{n!}}_{a_n} = \frac{2}{1}\cdot\frac{2}{2}\cdot\underbrace{\left[\frac{2}{3}\cdot\frac{2}{4}\cdots\frac{2}{n-1}\right]}_{<1}\cdot\frac{2}{n} < \frac{6}{n} = c_n$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$$

$\therefore$ $a_n \to 0$ $n \to \infty$ by the squeeze theorem.

Question 7. *(5 marks)* Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\frac{(-3)^{n+1}}{(2(n+1)+1)!}}{\frac{(-3)^n}{(2n+1)!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-3)^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-3)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(-3)^n}(-3)}{\cancel{(2n+1)!}(2n+2)(2n+3)} \cdot \frac{\cancel{(2n+1)!}}{\cancel{(-3)^n}} \right|$$

$$= \lim_{n\to\infty} \frac{3}{(2n+2)(2n+3)}$$

$$= 0 < 1$$

$\therefore$ the series converges absolutely by the ratio test.

Question 8. *(5 marks)* Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$$

Lets verify for absolute convergence

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n \ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Let $f(x) = \dfrac{1}{x \ln x}$

- $f(x)$ is positive for $x > 2$.
- $f(x)$ is continivous for $x > 2$.
- $f'(x) = \dfrac{-1}{(x \ln x)^2} \left(\ln x + \cancel{x} \dfrac{1}{\cancel{x}} \right) < 0$ for $x > 2$

$\therefore$ $f(x)$ is decreasing when $x > 2$.

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{b \to \infty} \int_2^{b} \frac{1}{x \ln x}\, dx = \lim_{b \to \infty} \int_{\ln 2}^{\ln b} \frac{1}{u}\, du$$

$u = \ln x$

$du = \dfrac{1}{x} dx$

$u(b) = \ln b$

$u(2) = \ln 2$

$$= \lim_{b \to \infty} \Big[\ln |u| \Big]_{\ln 2}^{\ln b}$$

$$= \lim_{b \to \infty} \ln |\overset{\infty}{\ln b}| - \ln \ln 2 = \infty$$

$\therefore$ the series diverges since the integral diverges, using the integral test.

$\therefore$ not absolutely convergent

Since $a_n = \dfrac{1}{n \ln n}$ decreases, $a_n > a_{n+1}$ and $\lim_{n \to \infty} a_n = 0$,

$\sum_{n=2}^{\infty} (-1)^n$ is convergent by the alternating series test.

$\therefore$ $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ is conditionally convergent.

Bonus Question. *(3 marks)*

$$\sum_{n=1}^{\infty}\int_{n}^{n+1} f(x)\,dx = 0$$

$$= \sum_{n=1}^{\infty}\left[\int_{n}^{0} f(x)\,dx + \int_{0}^{n+1} f(x)\,dx\right]$$

$$= \sum_{n=1}^{\infty}\left[-\int_{0}^{n} f(x)\,dx + \int_{0}^{n+1} f(x)\,dx\right] = \sum_{n=1}^{\infty}\left[\int_{0}^{n+1} f(x)\,dx - \int_{0}^{n} f(x)\,dx\right]$$

$$S_n = a_1 + a_2 + a_3 + \dots + a_{n-3} + a_{n-1} + a_n$$

$$= \left[\cancel{\int_{0}^{2} f(x)\,dx} - \int_{0}^{1} f(x)\,dx\right] + \left[\cancel{\int_{0}^{3} f(x)\,dx} - \cancel{\int_{0}^{2} f(x)\,dx}\right] + \left[\int_{0}^{4} f(x)\,dx - \cancel{\int_{0}^{3} f(x)\,dx}\right]$$

$$+ \dots +$$

$$\left[\cancel{\int_{0}^{n-1} f(x)\,dx} - \int_{0}^{n-2} f(x)\,dx\right] + \left[\cancel{\int_{0}^{n} f(x)\,dx} - \cancel{\int_{0}^{n-1} f(x)\,dx}\right] + \left[\int_{0}^{n+1} f(x)\,dx - \cancel{\int_{0}^{n} f(x)\,dx}\right]$$

$$= -\int_{0}^{1} f(x)\,dx + \int_{0}^{n+1} f(x)\,dx$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(-\int_{0}^{1} f(x)\,dx + \int_{0}^{n+1} f(x)\,dx\right)$$

$$= -\int_{0}^{1} f(x)\,dx + \lim_{n\to\infty}\int_{0}^{n+1} f(x)\,dx$$

$$= -\int_{0}^{1} f(x)\,dx + \int_{0}^{1} f(x)\,dx$$

$$= 0$$

<u>Prove:</u> If $\int_{0}^{1} f(x)\,dx = \lim_{b\to\infty}\int_{0}^{b} f(x)\,dx$ then