Dawson College: **Linear Algebra**: 201-NYC-05-S04: Fall 2014

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Test 1

This test is graded out of 47 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1.

a. *(6 marks)* Solve the following system by Gauss-Jordan elimination:

$$\begin{array}{rcrcrcrcrcl} 3x_1 & - & 2x_2 & + & x_3 & - & 3x_4 & + & x_5 & = & 0 \\ 4x_1 & + & 3x_2 & - & x_3 & + & x_4 & - & 2x_5 & = & 0 \\ 7x_1 & + & x_2 & & & - & 2x_4 & - & x_5 & = & 0 \\ & & & & x_3 & - & x_4 & & & = & 0 \end{array}$$

b. *(2 marks)* Find two particular nontrivial solution to the above system.

$$\begin{bmatrix} 3 & -2 & 1 & -3 & 1 & 0 \\ 4 & 3 & -1 & 1 & -2 & 0 \\ 7 & 1 & 0 & -2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{l} \sim 3R_2 \to R_2 \\ \quad 3R_3 \to R_3 \end{array} \begin{bmatrix} 3 & -2 & 1 & -3 & 1 & 0 \\ 12 & 9 & -3 & 3 & -6 & 0 \\ 21 & 3 & 0 & -6 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{l} \sim -4R_1+R_2 \to R_2 \\ \quad -7R_1+R_3 \to R_3 \end{array} \begin{bmatrix} 3 & -2 & 1 & -3 & 1 & 0 \\ 0 & 17 & -7 & 15 & -10 & 0 \\ 0 & 17 & -7 & 15 & -10 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

$$\sim -R_2+R_3 \to R_3 \begin{bmatrix} 3 & -2 & 1 & -3 & 1 & 0 \\ 0 & 17 & -7 & 15 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

$$\sim R_3 \leftrightarrow R_4 \begin{bmatrix} 3 & -2 & 1 & -3 & 1 & 0 \\ 0 & 17 & -7 & 15 & -10 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{l} -R_3+R_1 \to R_1 \\ \sim 7R_3+R_2 \to R_2 \end{array} \begin{bmatrix} 3 & -2 & 0 & -2 & 1 & 0 \\ 0 & 17 & 0 & 8 & -10 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim 17R_1 \to R_1 \begin{bmatrix} 51 & -34 & 0 & -34 & 17 & 0 \\ 0 & 17 & 0 & 8 & -10 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim 2R_2+R_1 \to R_1 \begin{bmatrix} 51 & 0 & 0 & -18 & -3 & 0 \\ 0 & 17 & 0 & 8 & -10 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{l} \sim \frac{1}{51}R_1 \to R_1 \\ \quad \frac{1}{17}R_2 \to R_2 \end{array} \begin{bmatrix} 1 & 0 & 0 & -\frac{18}{51} & -\frac{3}{51} & 0 \\ 0 & 1 & 0 & 8/17 & -10/17 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_4 & x_5 are free var.

$x_4 = s \quad s,t \in \mathbb{R}$

$x_5 = t$

$\therefore x = (x_1, x_2, x_3, x_4, x_5)$

$= \left(\frac{6}{17}s + \frac{1}{17}t,\ -\frac{8}{17}s + \frac{10}{17}t,\ s,\ s,\ t\right)$

b) $s=0,\ t=1 \quad x = \left(\frac{1}{17}, \frac{10}{17}, 0, 0, 1\right)$

$s=1,\ t=0 \quad x = \left(\frac{6}{17}, \frac{-8}{17}, 1, 1, 0\right)$

Question 2. Consider the matrices:

$$A=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&2\end{bmatrix},\ B=\begin{bmatrix}2&2&1\\0&-4&0\end{bmatrix}\ C=\begin{bmatrix}0&1\\-3&0\\3&-1\end{bmatrix}\ D=\begin{bmatrix}0&1\\1&-2\end{bmatrix},\ F=\begin{bmatrix}1&-1\\0&-2\\1&-3\end{bmatrix}$$

a. (2 marks) Compute the following, if possible.

tr(BC)

$$tr\left(\begin{bmatrix}2&2&1\\0&-4&0\end{bmatrix}\begin{bmatrix}0&1\\-3&0\\3&-1\end{bmatrix}\right)=tr\left(\begin{bmatrix}-3&1\\12&0\end{bmatrix}\right)$$

$$=-3+0=-3$$

b. *(2 marks)* Compute the following, if possible.

A^{-2}

A is invertible since the elements of the main diagonal are non zero. So

$$A^{-2}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1/4\end{bmatrix}$$

c. *(6 marks)* Find E, if possible.

$(\text{tr}(D)D-E^t)^{-1}=BA^{-2}F$

$$\left[(tr(D)D-E^t)^{-1}\right]^{-1}=\left[BA^{-2}F\right]^{-1}$$

$$tr(D)D-E^t=\left[BA^{-2}F\right]^{-1}$$

$$E^t=tr(D)D-\left[BA^{-2}F\right]^{-1}$$

$$E=\left(tr(D)D-\left[BA^{-2}F\right]^{-1}\right)^t$$

So $tr(D)=-2$

$$BA^{-2}F=\begin{bmatrix}2&2&1\\0&-4&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1/4\end{bmatrix}\begin{bmatrix}1&-1\\0&-2\\1&-3\end{bmatrix}=\begin{bmatrix}2&2&1\\0&-4&0\end{bmatrix}\begin{bmatrix}1&-1\\0&-2\\1/4&-3/4\end{bmatrix}=\begin{bmatrix}9/4&-27/4\\0&8\end{bmatrix}$$

$$\left[BA^{-2}F\right]^{-1}=\frac{1}{18}\begin{bmatrix}8&27/4\\0&9/4\end{bmatrix}=\begin{bmatrix}4/9&3/8\\0&1/8\end{bmatrix}$$

$$E=\left(-2\begin{bmatrix}0&1\\1&-2\end{bmatrix}-\begin{bmatrix}4/9&3/8\\0&1/8\end{bmatrix}\right)^t$$

$$=\begin{bmatrix}-4/9&-19/8\\-2&31/8\end{bmatrix}^t$$

$$=\begin{bmatrix}-4/9&-2\\-19/8&31/8\end{bmatrix}$$

Question 3. Consider

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

a. (4 marks) Find A^{-1}.

b. (2 marks) Write A^{-1} as a product of elementary matrices.

c. (2 marks) Using a. solve $Ax = b$ where $x = [x_1, x_2, x_3]^t$ and $b = [1, -2, 1]^t$.

a) $[A \mid I]$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\sim_{\frac{1}{3}R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right]$$

$$\sim_{\substack{-3R_3+R_1 \to R_1 \\ -2R_3+R_2 \to R_2}} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right]$$

$$\sim_{-2R_2+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & \frac{1}{3} \\ 0 & 1 & 0 & 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right]$$

$\therefore$ A is invertible $A^{-1} = \begin{bmatrix} 1 & -2 & \frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

c) $Ax = b$

$A^{-1}Ax = A^{-1}b$

$Ix = A^{-1}b$

$x = A^{-1}b$

$= \begin{bmatrix} 1 & -2 & \frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 16/3 \\ -8/3 \\ 1/3 \end{bmatrix}$

b) $E_4 E_3 E_2 E_1 A = I$

So $A^{-1} = E_4 E_3 E_2 E_1$

where

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

$E_2 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E_4 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Question 4.[1]

a. *(2 marks)* Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Find a 2×2 matrix B such that $AB = \mathbf{0}$ but $BA \neq \mathbf{0}$.

b. Let A and B be non-zero $n \times n$ matrices such that $AB = \mathbf{0}$ but $BA \neq \mathbf{0}$.

i. *(2 marks)* Prove that $(BA)^2 = \mathbf{0}$

ii. *(2 marks)* Prove that B is singular.

a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = 0$

$b_{11} + b_{21} = 0 \qquad b_{11} = -b_{21}$

$b_{12} + b_{22} = 0 \qquad b_{12} = -b_{22}$

$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq 0$

$b_{11} + b_{12} \neq 0$

or $b_{21} + b_{22} \neq 0$

So let $b_{11} = 1$

$b_{22} = 2$

$B = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$ ✓

bi) $(BA)^2 = BABA$

$= B0A$ from premise $AB = 0$

$= 0$

ii) Suppose that B is invertible.

$BB^{-1} = I$

$ABB^{-1} = AI$

$0B^{-1} = A$

$0 = A$ ↯ A is nonzero

$\therefore$ B is not invertible

$\therefore$ B is singular.

Question 5.[2] *(4 marks)* Let

$$A = \begin{bmatrix} 2 & 1 \\ x & y \end{bmatrix}$$

Find x and y such that A^2 is symmetric, if possible.

$A^2 = AA$

$= \begin{bmatrix} 2 & 1 \\ x & y \end{bmatrix}\begin{bmatrix} 2 & 1 \\ x & y \end{bmatrix}$

$= \begin{bmatrix} 4+x & 2+y \\ 2x+xy & x+y^2 \end{bmatrix}$

$(A^2)^T = A^2$

$\begin{bmatrix} 4+x & 2x+xy \\ 2+y & x+y^2 \end{bmatrix} = \begin{bmatrix} 4+x & 2+y \\ 2x+xy & x+y^2 \end{bmatrix}$

So $2+y = 2x + xy$

$0 = 2x + xy - 2 - y$

$0 = x(2+y) - (2+y)$

$0 = (x-1)(2+y)$

$x = 1 \qquad y = -2$

So if $x = 1$ then $y \in \mathbb{R}$

and if $y = -2$ then $x \in \mathbb{R}$

[1]From a John Abbott Final Examination

[2]From a John Abbott Final Examination

Question 6.[3] A matrix X is called a *weak generalized inverse* of A if $AXA = A$

a. *(3 marks)* For what value of k is $\begin{bmatrix} k & k \\ k & k \\ k & k \end{bmatrix}$ a weak generalized inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

b. *(2 marks)* Show that if the system $A\mathbf{x} = \mathbf{b}$ is consistent then $X\mathbf{b}$ will be a solution to this system

$$AXA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} K & K \\ K & K \\ K & K \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} K+K & K+K & K+K \\ K+K & K+K & K+K \\ K+K & K+K & K+K \end{bmatrix}$$

$$= \begin{bmatrix} (K+K)+(K+K)+(K+K) & (K+K)+(K+K)+(K+K) & (K+K)+(K+K)+(K+K) \\ (K+K)+(K+K)+(K+K) & (K+K)+(K+K)+(K+K) & (K+K)+(K+K)+(K+K) \end{bmatrix}$$

$$= \begin{bmatrix} 6K & 6K & 6K \\ 6K & 6K & 6K \end{bmatrix}$$

So $AXA = A$
if $K = 1/6$

b) AXb
$= AXAx_1$
$= Ax_1$
$= b$

Since $Ax = b$ is consistent
$\exists x_1$ s.t. $Ax_1 = b$

since $A = AXA$

$\therefore$ Xb is a solution to $Ax = b$.

[3]From a John Abbott Final Examination

Question 7.(*6 marks*) Consider the following system:

$$\begin{array}{rcrcrcl} x & + & y & + & 7z & = & c_1 \\ x & + & 2y & + & 3z & = & c_2 \\ 2x & + & 3y & + & 10z & = & 1 \end{array}$$

and

$$\begin{array}{rcrcrcl} 2x & + & y & - & 7z & = & c_1 \\ 3x & - & 2y & + & 3z & = & c_2 \\ 7x & & & - & 11z & = & 0 \end{array}$$

. If any, for what value(s) of c_1, c_2 are both system consistent.

$$\begin{bmatrix} 1 & 1 & 7 & C_1 \\ 1 & 2 & 3 & C_2 \\ 2 & 3 & 10 & 1 \end{bmatrix}$$

$$\sim \begin{array}{l} -R_1+R_2 \to R_2 \\ -2R_1+R_3 \to R_3 \end{array} \begin{bmatrix} 1 & 1 & 7 & C_1 \\ 0 & 1 & -4 & C_2-C_1 \\ 0 & 1 & -4 & 1-2C_1 \end{bmatrix}$$

$$\sim -R_2+R_3 \to R_3 \begin{bmatrix} 1 & 1 & 7 & C_1 \\ 0 & 1 & -4 & C_2-C_1 \\ 0 & 0 & 0 & -C_2-C_1+1 \end{bmatrix}$$

this system is consistent if $-C_2 - C_1 + 1 = 0$

$C_1 + C_2 = 1$

$C_1 = -C_2 + 1$

$$\begin{bmatrix} 2 & 1 & -7 & C_1 \\ 3 & -2 & 3 & C_2 \\ 7 & 0 & -11 & 0 \end{bmatrix}$$

$$\sim \begin{array}{l} 2R_2 \to R_2 \\ 2R_3 \to R_3 \end{array} \begin{bmatrix} 2 & 1 & -7 & C_1 \\ 6 & -4 & 6 & 2C_2 \\ 14 & 0 & -22 & 0 \end{bmatrix}$$

$$\sim \begin{array}{l} -3R_1+R_2 \to R_2 \\ -7R_1+R_3 \to R_3 \end{array} \begin{bmatrix} 2 & 1 & -7 & C_1 \\ 0 & -7 & 27 & 2C_2-3C_1 \\ 0 & -7 & 27 & -7C_1 \end{bmatrix}$$

$$\sim -R_2+R_3 \to R_3 \begin{bmatrix} 2 & 1 & -7 & C_1 \\ 0 & -7 & 27 & 2C_2-3C_1 \\ 0 & 0 & 0 & -4C_1-2C_2 \end{bmatrix}$$

this system is consistent if $-4C_1 - 2C_2 = 0$

$4C_1 + 2C_2 = 0$

$\therefore$ both system are consistent if

$4(-C_2+1) + 2C_2 = 0$

$-4C_2 + 4 + 2C_2 = 0$

$4 = 2C_2$

$C_2 = 2$

and $C_1 = -2 + 1 = -1$

Bonus Question. *(5 marks)* If A is 2×2, show that $A^{-1}=A^T$ if and only if $A=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$ for some θ or $A=\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$.

for some θ.

[$\Leftarrow$]

Let $A=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$ then $A^{-1}=\dfrac{1}{\cos^2\theta+\sin^2\theta}\begin{bmatrix}\cos\theta & -\sin\theta\\ +\sin\theta & \cos\theta\end{bmatrix}$

$=\dfrac{1}{1}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

$\therefore A^{-1}=A^T$

Let $A=\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$ then $A^{-1}=\dfrac{1}{-\cos^2\theta-\sin^2\theta}\begin{bmatrix}-\cos\theta & -\sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$

$=-\begin{bmatrix}-\cos\theta & -\sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$

$=\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$

[$\Rightarrow$] $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, $A^T=\begin{bmatrix}a & c\\ b & d\end{bmatrix}$

$A^{-1}=\dfrac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$

$A^{-1}=A^T$

$\dfrac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}=\begin{bmatrix}a & c\\ b & d\end{bmatrix}$

let $K=ad-bc$

$\begin{bmatrix}d/K & -b/K\\ -c/K & a/K\end{bmatrix}=\begin{bmatrix}a & c\\ b & d\end{bmatrix}$

(1) $\dfrac{d}{K}=a \Leftrightarrow d=aK$ (3)

(4) $\dfrac{a}{K}=d \Leftrightarrow a=dK$ (2)

By (1),(2),(3),(4) it follows that

$K=\pm 1$

if $K=1$ then $d=a$ and $-b=c$

So we have

$1=ad-bc$

$1=a^2+b^2$

Let $a=\cos\theta$

$b=\sin\theta$

and we get

$A=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$

if $K=-1$

we get

$A=\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$.