**Dawson College: Linear Algebra:** 201-NYC-05-S04: Fall 2014

**Name:** Y. Lamontagne
**Student ID:**

# Test 1

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

**Question 1.** Given

$$\begin{array}{lrcll}
\mathscr{L}_1: & (x,y,z) & = & (2+5t, 1+t, -t) & t\in\mathbb{R} \\
\mathscr{L}_2: & (x,y,z) & = & (7+2t, 4, 10t) & t\in\mathbb{R} \\
\mathscr{L}_3: & (x,y,z) & = & (9-t, 2, 9-5t) & t\in\mathbb{R} \\
\mathscr{P}_1: & x-2y+3z-11 & = & 0 & \\
\mathscr{P}_2: & -5x-y+z+31 & = & 0 & \\
\mathscr{P}_3: & -3x+6y-9z+1 & = & 0 &
\end{array}$$

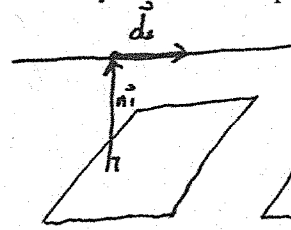


fig 1.

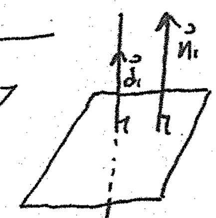


fig. 2.

a. *(2 marks)* Are $\mathscr{P}_1$ and $\mathscr{L}_2$ parallel, perpendicular, or neither, justify?

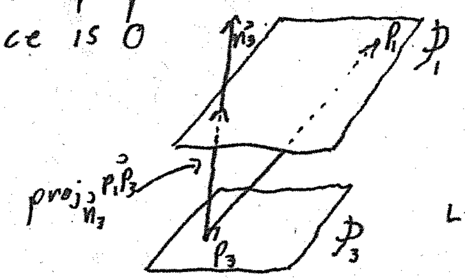
b. *(5 marks)* Are $\mathscr{P}_1$ and $\mathscr{P}_3$ parallel, perpendicular, or neither, justify? If parallel, find the shortest distance between the two planes using projections.

c. *(3 marks)* Are $\mathscr{P}_2$ and $\mathscr{P}_3$ parallel, perpendicular, or neither, justify? Find the shortest distance between the two planes, justify.

d. *(5 marks)* Are $\mathscr{L}_2$ and $\mathscr{L}_3$ parallel, perpendicular, or neither, justify? If parallel, find the shortest distance between the two lines using projections.

a) $\vec{n}_1 = (1,-2,3)$, $\vec{d}_2 = (2,0,10)$, $\vec{n}_1\cdot\vec{d}_2 \neq 0$ ∴ not parallel, see fig. 1
$\vec{n}_1 \neq k\vec{d}_2$ ∴ not perpendicular, see fig. 2.

c) $\vec{n}_2 = (-5,-1,1)$, $\vec{n}_3 = (-3,6,-9)$, $\vec{n}_2\cdot\vec{n}_3 = 0$ ∴ perpendicular therefore the two planes intersect, so the distance is 0




Let $y=z=0$
$x=11$
$P_1(11,0,0)$
Let $y=z=0$
$x=\frac{1}{3}$
$P_3(\frac{1}{3},0,0)$

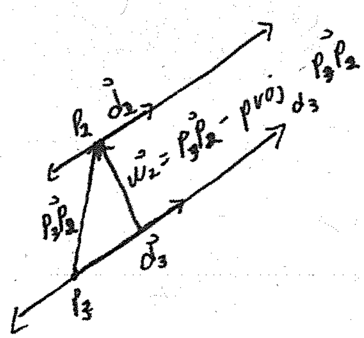
b) $\vec{n}_1 = (1,-2,3)$, $\vec{n}_3 = (-3,6,-9)$, $\vec{n}_3 = -3\vec{n}_1$

$$\vec{P_1P_3} = P_3 - P_1 = \left(\tfrac{1}{3},0,0\right)-(11,0,0) = \left(-\tfrac{32}{3},0,0\right)$$

$$\text{proj}_{\vec{n}_3}\vec{P_1P_3} = \frac{\vec{n}_3\cdot\vec{P_1P_3}}{\vec{n}_3\cdot\vec{n}_3}\vec{n}_3 = \frac{+32}{9+36+81}(-3,6,-9)$$

$$\text{distance} = \|\text{proj}_{\vec{n}_3}\vec{P_1P_3}\| = \left\|\frac{32}{126}(-3,6,-9)\right\| = \frac{32}{126}\sqrt{126} = \frac{16}{63}\sqrt{126} = \frac{16}{21}\sqrt{14}$$

d) $\vec{d}_2 = (2,0,10)$, $\vec{d}_3 = (-1,0,-5)$, $\vec{d}_2 = -2\vec{d}_3$ ∴ parallel




$$\vec{w}_2 = \vec{P_3P_2} - \text{proj}_{\vec{d}_3}\vec{P_3P_2}$$

$$= (-2,2,-9) - \frac{(-1,0,-5)\cdot(-2,2,-9)}{(-1,0,-5)\cdot(-1,0,-5)}(-1,0,-5)$$

$$= (-2,2,-9) - \frac{2+45}{1+25}(-1,0,-5)$$

$$= (-2,2,-9) - \frac{47}{26}(-1,0,-5)$$

$$= \left(\frac{-5}{26}, 2, \frac{1}{26}\right)$$

$$\vec{P_3P_2} = P_2 - P_3 = (7,4,0)-(9,2,9) = (-2,2,-9)$$

$$\text{distance} = \|\vec{w}_2\| = \frac{\sqrt{(-5)^2+52^2+1^2}}{26} = \frac{\sqrt{2730}}{26}$$

**Question 2.** Given

$$A = \begin{bmatrix} 9 & 9 & 1 & 1 & 9 & 3 & 4 & 1 & 0 & 9 \\ 9 & 9 & 1 & 1 & 4 & 9 & 2 & 7 & 7 & 9 \\ 0 & 0 & 7 & 1 & 1 & 4 & 9 & 2 & 7 & 7 \\ 0 & 0 & 0 & 7 & 1 & 1 & 4 & 9 & 2 & 7 \\ 0 & 0 & 0 & 0 & 2 & 1 & 1 & 4 & 9 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 4 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 3 & 2 & 6 \\ 0 & 2 & 3 & -3 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -12 \end{bmatrix}, \quad C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad D = \begin{bmatrix} b & d \\ 3a-2b & 3c-2d \end{bmatrix}$$

a. *(4 marks)* If $F$ is a $10 \times 10$ matrix show that $AF$ is not invertible.

b. *(4 marks)* If $E$ is an invertible matrix then evaluate $\det(\det(E)\,\mathrm{adj}(B))$, justify fully.

c. *(4 marks)* If $\det(D) = 2$ then determine $\det(C)$.

a) $A \sim \begin{matrix} -R_1+R_2 \to R_2 \\ -R_9+R_{10} \to R_{10} \end{matrix}$ $G$ then $\det G = 9(0)(7)(7)(2)(1)(4)(1)(1)(1) = 0$ $\therefore \det A = 0$

and $\det(AF) = \det(A)\det(F) = 0 \det F = 0$ $\therefore$ $AF$ is not invertible

b) $\frac{1}{(\det(E))^4} \det(\det(E)\,\mathrm{adj}(B)) = \frac{1}{\cancel{(\det(E))^4}} \cancel{(\det(E))^4} \det(\mathrm{adj}(B))$ since adj B is $4 \times 4$

$= \det(\mathrm{adj}\, B)$

$= (\det B)^{4-1} = (-2(2)(1)(-12))^3 = 48^3$

c) $|C| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

$= -\begin{vmatrix} b & d \\ a & c \end{vmatrix}$

$= -\frac{1}{3}\begin{vmatrix} b & d \\ 3a & 3c \end{vmatrix}$

$= -\frac{1}{3}\begin{vmatrix} b & d \\ 3a-2b & 3c-2d \end{vmatrix}$ $-2R_1+R_2 \to R_2$ does not change the det.

$= -\frac{1}{3} 2$

$= -\frac{2}{3}$

**Question 3.** If $A$ is an $n \times n$ matrix which entries are all divisible by 2.

a. *(2 marks)* Justify that the entries of $\operatorname{adj}(A)$ are all divisible by 2.

b. *(2 marks)* Prove using a. that that the entries of $A^{-1}$ are all integers.

a) The adjoint is the transpose of the matrix of cofactors.
A cofactor is obtained by multiplying entries and subtracting entries.
If all the entries are divisible by 2 then the product and difference is divisible by 2. Hence the entries of adj (A) are divisible by 2.

b) Since $A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{2} \operatorname{adj} A$ and all entries of $A$ are divisible by 2

So $A^{-1}$ are all integers.

**Question 4.** *(2 marks)* If $\vec{u}, \vec{v}$ and $\vec{w}$ be pairwise orthogonal vectors and are all of the same length show that they all make the same angle with $\vec{u}+\vec{v}+\vec{w}$.

$\|\vec{u}\| = \|\vec{v}\| = \|\vec{w}\| = k$ and $\|\vec{u}+\vec{v}+\vec{w}\| = \ell$

For $\vec{u}$ & $\vec{u}+\vec{v}+\vec{w}$

$(\vec{u}+\vec{v}+\vec{w})\cdot\vec{u} = \|\vec{u}\|\|\vec{u}+\vec{v}+\vec{w}\| \cos\theta$

$\vec{u}\cdot\vec{u} + \underbrace{\vec{v}\cdot\vec{u}}_{0} + \underbrace{\vec{w}\cdot\vec{u}}_{0} = k\ell\cos\theta$

$\|\vec{u}\|^2 = k\ell\cos\theta$

$k^2 = k\ell\cos\theta$

$\frac{k}{\ell} = \cos\theta$

For $\vec{v}$ & $\vec{u}+\vec{v}+\vec{w}$

$(\vec{u}+\vec{v}+\vec{w})\cdot\vec{v} = \|\vec{v}\|\|\vec{u}+\vec{v}+\vec{w}\|\cos\theta$

$\underbrace{\vec{u}\cdot\vec{v}}_{0} + \vec{v}\cdot\vec{v} + \underbrace{\vec{w}\cdot\vec{v}}_{0} = k\ell\cos\theta$

$\vec{v}\cdot\vec{v} = k\ell\cos\theta$

$\|\vec{v}\|^2 = k\ell\cos\theta$

$k^2 = k\ell\cos\theta$

$\frac{k}{\ell} = \cos\theta$

For $\vec{w}$ & $\vec{u}+\vec{v}+\vec{w}$

$(\vec{u}+\vec{v}+\vec{w})\cdot\vec{w} = \|\vec{u}\|\|\vec{u}+\vec{v}+\vec{w}\|\cos\theta$

$\underbrace{\vec{u}\cdot\vec{w}}_{0} + \underbrace{\vec{v}\cdot\vec{w}}_{0} + \vec{w}\cdot\vec{w} = k\ell\cos\theta$

$\|\vec{w}\|^2 = k\ell\cos\theta$

$k^2 = k\ell\cos\theta$

$\frac{k}{\ell} = \cos\theta$

**Question 5.** *(2 marks)* Prove or disprove: The general solution of the nonhomogeneous linear system $Ax = b$ can be obtaines by adding $b$ to the general solution of the homogeneous linear system $Ax = 0$.

<u>disprove:</u>

Let
$$\begin{aligned} x \quad &= 1 \\ x + y &= 1 \end{aligned}$$

then
$$\begin{aligned} x \quad &= 0 \\ x + y &= 0 \end{aligned}$$

has a unique solution $(x,y) = (0,0)$

and $(0,0) + b = (0,0) + (1,1) = (1,1)$ is not a solution of the system $Ax = b$.

**Question 6.** *(4 marks)* Solve *only* for $x_2$ using Cramer's rule.

$$\begin{array}{rcrcrcr} 3x_1 & - & x_2 & + & x_3 & = & 1 \\ x_1 & + & 3x_2 & - & 3x_3 & = & -3 \\ x_1 & - & 2x_2 & - & x_3 & = & -1 \end{array}$$

$$X_2 = \frac{\det A_2}{\det A} = \frac{0}{-30} = 0$$

$$|A| = \left|\begin{array}{ccc|cc} 3 & -1 & 1 & 3 & -1 \\ 1 & 3 & -3 & 1 & 3 \\ 1 & -2 & -1 & 1 & -2 \end{array}\right| = 3(3)(-1) + \cancel{(-1)(-3)(1)} + 1(1)(-2)$$

$$- \cancel{(1)(3)(1)} - (3)(-3)(-2) - (-1)(1)(-1)$$

$$= -30$$

$$|A_2| = \begin{vmatrix} 3 & 1 & 1 \\ 1 & -3 & -3 \\ 1 & -1 & -1 \end{vmatrix} = 0 \quad \text{since} \quad C_2 = C_3$$

**Question 7.** *(3 marks)* Solve for $\lambda$.

$$\begin{vmatrix} \lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & \lambda & -6 \\ 1 & 3 & \lambda - 5 \end{vmatrix}$$

$$\lambda(1-\lambda) + 3 = \begin{vmatrix} \lambda & -6 \\ 3 & \lambda - 5 \end{vmatrix} - 3 \begin{vmatrix} 2 & \lambda \\ 1 & 3 \end{vmatrix}$$

$$\lambda - \lambda^2 + 3 = \lambda(\lambda - 5) + 18 - 3[6 - \lambda]$$

$$0 = \lambda^2 - \lambda - 3 + \lambda^2 - 5\lambda + 18 - 18 + 3\lambda$$

$$0 = 2\lambda^2 - 3\lambda - 3$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{3 \pm \sqrt{(-3)^2 - 4(2)(3)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{33}}{4}$$

**Bonus Question.** *(5 marks)*[1]
The Cayley-Hamilton theorem (named after the mathematicians Arthur Cayley and William Rowan Hamilton) states that every square matrix over a commutative ring (such as the real or complex field) satisfies its own characteristic equation.

More precisely, if $A$ is a given $n \times n$ matrix and $I_n$ is the $n \times n$ identity matrix, then the characteristic polynomial of $A$ is defined as

$$p(\lambda) = \det(\lambda I_n - A)$$

where det is the determinant operation. Since the entries of the matrix are (linear or constant) polynomials in $\lambda$, the determinant is also an $n^{th}$ order polynomial in $\lambda$.

The Cayley-Hamilton theorem states that "substituting" the matrix $A$ for $\lambda$ in this polynomial results in the zero matrix,

$$p(A) = 0$$

The powers of $A$, obtained by substitution from powers of $\lambda$, are defined by repeated matrix multiplication; the constant term of $p(\lambda)$ gives a multiple of the power $A^0$, which power is defined as the identity matrix.

Prove the Cayley-Hamilton theorem for $2 \times 2$ matrices.

$$p(\lambda) = \det\left[\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right] = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = (\lambda - a)(\lambda - d) - bc$$

$$= \lambda^2 - d\lambda - a\lambda + ad - bc$$

$$= \lambda^2 - (d+a)\lambda + (ad - bc)$$

$$p(A) = A^2 - (d+a)A + (ad+bc)I_2$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - (d+a)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad - bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{bmatrix} + \begin{bmatrix} -ad - a^2 & -db - ab \\ -dc - ac & -d^2 - ad \end{bmatrix} + \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

[1] Wikipedia contributors. "Cayley-Hamilton theorem." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 17 Oct. 2014. Web. 31 Oct. 2014.