Dawson College: **Linear Algebra**: 201-NYC-05-S04: Fall 2014

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Test 3

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Let $\mathscr{H} = \{A \mid A \text{ is a } 2\times 2 \text{ matrix and } A^T = -A\}$ with the usual addition and scalar multiplication.

a. *(2 marks)* Give an example of a non-zero matrix in $\mathscr{H}$. Justify.

b. *(2 marks)* Does $\mathscr{H}$ satisfy closure under vector addition? Justify.

c. *(2 marks)* Does $\mathscr{H}$ contain the zero vector of $\mathscr{M}_{2\times 2}$ (the vector space of 2×2 matrices)? Justify.

d. *(2 marks)* Does $\mathscr{H}$ satisfy closure under scalar multiplication? Justify.

e. *(2 marks)* Is $\mathscr{H}$ a vector subspace of $\mathscr{M}_{2\times 2}$ (the vector space of 2×2 matrices)? Justify.

a) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -A \quad \therefore A \in \mathcal{H}$

b) Let $A, B \in \mathcal{H}$ since $(A+B)^T = A^T + B^T$
$= -A - B$
$= -(A+B)$

c) $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{H}$ since $0^T = -0$

d) Let $k \in \mathbb{R}$, $A \in \mathcal{H}$ $kA \in \mathcal{H}$ since $(kA)^T = kA^T = k(-A) = -(kA)$

e) $\mathcal{H}$ is a subspace since it is closed under vector addition and scalar multiplication.

Question 2. *(2 marks)* Determine whether the following is a vector space:

$$\mathscr{Y} = \{(1,y)\ |y \in \mathbb{R}\}$$

with the following polynomial addition and scalar multiplication.

$$(1,y_1)+(1,y_2) = (1,y_1+y_2)$$

and

$$k(1,y) = (1,y)$$

Not a vector space since if $r,s \in \mathbb{R}$ and $\vec{y} \in \mathscr{Y}$ then $(r+s)\vec{y} \neq r\vec{y}+s\vec{y}$

$(r+s)\vec{y} = (r+s)(1,y) = (1,y)$

and

$r\vec{y} + s\vec{y} = r(1,y) + s(1,y) = (1,y)+(1,y) = (1,2y)$

Question 3. Let $\mathscr{W} = \{p(x) = a_0 + a_1x + a_2x^2 |\ p(1) = 0\}$ be a vector subspace of $\mathscr{P}_n$.

a. *(4 marks)* Find a basis S for $\mathscr{W}$.

b. *(2 marks)* Determine the dimension of $\mathscr{W}$, Justify.

c. *(2 marks)* Find the coordinate vector of $p(x) = -1-x+2x^2$ relative to the basis S.

a) $p(x) = a_0 + a_1 x + a_2 x^2 \in W$, $p(1) = 0$

$0 = a_0 + a_1 + a_2$

$a_0 = -a_1 - a_2$

So $p(x) = (-a_1 - a_2) + a_1 x + a_2 x^2 = a_1 \underbrace{(-1+x)}_{P_1} + a_2 \underbrace{(-1+x^2)}_{P_2}$

So $S = \{P_1, P_2\}$ spans W and S is linearly independent since P_1 is not a multiple of P_2. $\therefore$ S is a basis for W

b) $\dim W = 2$ since its basis has 2 elements

c) $-1 - x + 2x^2 = ap_1 + bp_2$

$= a(-1+x) + b(-1+x^2)$, so $a=-1$, $b=2$

$\therefore (p(x))_S = (-1, 2)$

Question 4. Let $\vec{u} = (1, 0, 2)$ and $\vec{v} = (1, -2, 3)$.

a. *(2 marks)* Find a vector $\vec{w}$ of length $\sqrt{21}$ orthogonal to $\vec{u}$ and $\vec{v}$

b. *(2 marks)* Compute the scalar triple product of $\vec{u}$, $\vec{v}$ and $\vec{w}$

c. *(2 marks)* Find the volume of the parallelepiped with sides $3\vec{u}, 2\vec{v}$ and $-\vec{w}$.

d. *(4 marks)* Let $\mathscr{P}$ be a plane that passes through the point $P_0(2, 1, -1)$ and is parallel to $\vec{u}$ and $\vec{v}$. Find the point on $\mathscr{P}$ closest to $P(3, 2, 1)$.

a) $\vec{u} \times \vec{v} = \left(\begin{vmatrix} 0 & -2 \\ 2 & 3 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} \right) = (4, -1, -2)$

$\begin{matrix} 1 & 1 \\ 0 & -2 \\ 2 & 3 \end{matrix}$

$\vec{w} = \sqrt{21} \dfrac{\vec{u} \times \vec{v}}{\|\vec{u} \times \vec{v}\|} = \sqrt{21} \dfrac{(4, -1, -2)}{\sqrt{4^2 + (-1)^2 + (-2)^2}} = \dfrac{\sqrt{21}}{\sqrt{21}} (4, -1, -2) = (4, -1, -2)$

b) $\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 1 & 0 & 2 \\ 1 & -2 & 3 \\ 4 & -1 & -2 \end{vmatrix} = 1 \cdot \begin{vmatrix} -2 & 3 \\ -1 & -2 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & -2 \\ 4 & -1 \end{vmatrix} = 7 + 2 \cdot 7 = 21$

c) $\text{vol} = \left| 3\vec{u} \cdot ((2\vec{v}) \times (-\vec{w})) \right| = |3 \cdot 2 \cdot (-1)| \, |\vec{u} \cdot (\vec{v} \times \vec{w})| = 6 \cdot (21) = 126$

d) equation of plane: $ax + by + cz = d$

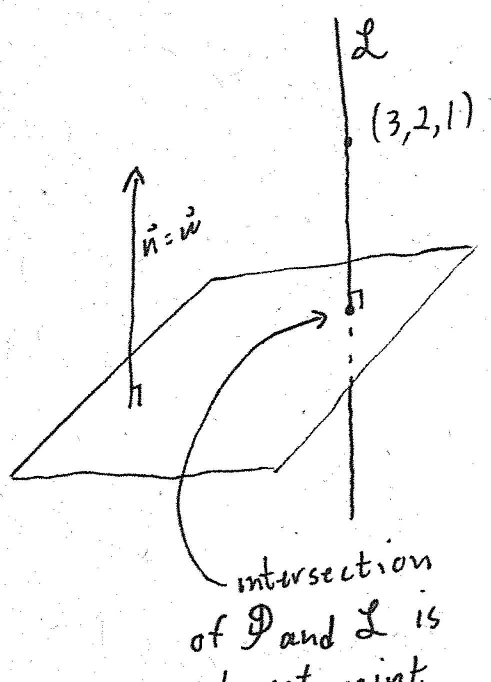
$4x - y - 2z = d$

solve for d by subbing P_0

$4(2) - 1 - 2(-1) = d$

$9 = d$

$\therefore 4x - y - 2z = 9$



$(x, y, z) = P + t\vec{d} \qquad t \in \mathbb{R}$

$= (3, 2, 1) + t(4, -1, -2)$

$= (3 + 4t, 2 - t, 1 - 2t)$

Intersection of $\mathscr{L}$ and $\mathscr{P}$

$4(3 + 4t) - (2 - t) - 2(1 - 2t) = 9$

$12 + 16t - 2 + t - 2 + 4t = 9$

$21t = 1$

$t = 1/21$

$\therefore (x, y, z) = \left(3 + 4\left(\frac{1}{21}\right), 2 - \frac{1}{21}, 1 - 2\left(\frac{1}{21}\right)\right)$

$= \left(\frac{67}{21}, \frac{41}{21}, \frac{19}{21}\right)$

Question 5. [1] Let $\vec{u}_1 = (\lambda,\lambda,2)$, $\vec{u}_2 = (\lambda,2,\lambda)$ and $\vec{u}_3 = (1,\lambda,-\lambda)$

a. *(5 marks)* For what value(s) of λ if any, is $\{\vec{u}_1,\vec{u}_2,\vec{u}_3\}$ linearly independent? If $\lambda = 1$ is one of those values, express $(1,2,3)$ as a linear combibation of $\{\vec{u}_1,\vec{u}_2,\vec{u}_3\}$ when $\lambda = 1$.

b. *(3 marks)* For what value(s) of λ if any, is $\{\vec{u}_2,\vec{u}_3\}$ linearly dependent? What geometrical object does span$\{\vec{u}_2,\vec{u}_3\}$ generates?

c. *(2 marks)* For what value(s) of λ if any, is span$\{\vec{u}_1,\vec{u}_2\}$ a line in $\mathbb{R}^3$.

a) $\vec{0} = K_1\vec{u}_1 + K_2\vec{u}_2 + K_3\vec{u}_3$

$(0,0,0) = K_1(\lambda,\lambda,2) + K_2(\lambda,2,\lambda) + K_3(1,\lambda,-\lambda)$

$$\underbrace{\begin{bmatrix} \lambda & \lambda & 1 \\ \lambda & 2 & \lambda \\ 2 & \lambda & -\lambda \end{bmatrix}}_{A}\begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$|A| \neq 0$ iff $\{\vec{u}_1,\vec{u}_2,\vec{u}_3\}$ is linearly independent

$$|A| = \begin{vmatrix} \lambda & \lambda & 1 \\ \lambda & 2 & \lambda \\ 2 & \lambda & -\lambda \end{vmatrix}\begin{matrix} \lambda & \lambda \\ \lambda & 2 \\ 2 & \lambda \end{matrix} = -2\lambda^2 + 2\lambda^2 + \lambda^2 - 4 - \lambda^3 + \lambda^3 = \lambda^2 - 4 \neq 0$$

$(\lambda-2)(\lambda+2) \neq 0$

$\lambda \neq 2 \qquad \lambda \neq -2$

$\therefore$ linearly independent iff $\lambda \neq 2, \lambda \neq -2$

$(1,2,3) = K_1(1,1,2) + K_2(1,2,1) + K_3(1,1,-1)$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & -1 & 3 \end{array}\right] \sim \begin{matrix} -R_1+R_2\to R_2 \\ -2R_1+R_3\to R_3 \end{matrix} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & -3 & 1 \end{array}\right] \sim_{R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & 2 \end{array}\right]$$

$\therefore K_2 = 1,\ K_3 = -2/3 \qquad K_1 = 1 - K_2 - K_3 = 1 - 1 - \left(\frac{-2}{3}\right) = 2/3$

b) $\{\vec{u}_2,\vec{u}_3\}$ is linearly dependent iff $\vec{u}_2$ is a multiple of $\vec{u}_3$

$\vec{u}_2 = K\vec{u}_3$

$(\lambda,2,\lambda) = K(1,\lambda,-\lambda)$

$\lambda = K$

$2 = \lambda K \iff 2 = \lambda^2$

$\lambda = K(-\lambda) \iff \lambda = -\lambda^2 \iff \lambda^2 + \lambda = 0$

$\lambda(\lambda+1) = 0$

$\lambda = 0 \quad \lambda = -1$

$\not y$

$\therefore$ not a multiple $\therefore$ linearly independent

$\therefore$ span$\{\vec{u}_2,\vec{u}_3\}$ generates a plane in $\mathbb{R}^3$

c) span$\{\vec{u}_1,\vec{u}_2\}$ generates a line if $\vec{u}_1$ is a parallel to $\vec{u}_2$. That is if $\lambda = 2$ then $\vec{u}_1 = K\vec{u}_2$.

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[1] Modified from a John Abbott Final Examination

Bonus Question. *(5 marks)* If U and W are subspaces of a vector space V then

$$U \cap W = \{\vec{v} \mid \vec{v} \in U \text{ and } \vec{v} \in W\}.$$

is a subspace of V.

Show that $U \cap W = \{\vec{0}\}$ if and only if $\{\vec{u}, \vec{v}\}$ is linearly independent for any nonzero vectors $\vec{u} \in U$ and $\vec{v} \in W$

[$\Rightarrow$] Suppose $\exists \vec{u} \neq \vec{0} \in U$ and $\exists \vec{v} \neq \vec{0} \in W$ such that $\{\vec{u}, \vec{v}\}$ is linearly dependent. Then $\exists k$ such that $\vec{u} = k\vec{v}$. It follows that $\vec{u} \in$ since W is closed under scalar multiplication. So $\vec{u} \in U \cap W$. ↯ since $U \cap W = \{\vec{0}\}$

$\therefore$ $\{\vec{u}, \vec{v}\}$ is linearly independent for any nonzero vectors $\vec{u} \in U$ and $\vec{v} \in W$

[$\Leftarrow$] Suppose $\vec{u} \neq 0$ and $\vec{u} \in U \cap W$.
Then $\vec{u} \in U$ and $\vec{u} \in W$ then $\{\vec{u}, \vec{u}\}$ is linearly dependent. ↯

$U \cap W = \{\vec{0}\}$