Dawson College: **Linear Algebra**: 201-NYC-05-S05: Fall 2014

Name: Y. Lamontagne
Student ID: ____________

Test 1

This test is graded out of ~~55~~ 44 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1.

a. *(6 marks)* Solve the following system by Gauss-Jordan elimination:

$$\begin{array}{rcrcrcrcr} 2x_1 & - & x_2 & + & x_3 & - & 3x_4 & = & 3 \\ 3x_1 & + & 3x_2 & - & x_3 & + & x_4 & = & 5 \\ 5x_1 & + & 2x_2 & & & - & 2x_4 & = & 8 \\ & & & & x_3 & - & x_4 & = & 1 \end{array}$$

b. *(2 marks)* Find two particular solution to the above system.

$$\left[\begin{array}{cccc|c} 2 & -1 & 1 & -3 & 3 \\ 3 & 3 & -1 & 1 & 5 \\ 5 & 2 & 0 & -2 & 8 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

$$\sim \begin{array}{l} 2R_2 \to R_2 \\ 2R_3 \to R_3 \end{array} \left[\begin{array}{cccc|c} 2 & -1 & 1 & -3 & 3 \\ 6 & 6 & -2 & 2 & 10 \\ 10 & 4 & 0 & -4 & 16 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

$$\sim \begin{array}{l} -3R_1 + R_2 \to R_2 \\ -5R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{cccc|c} 2 & -1 & 1 & -3 & 3 \\ 0 & 9 & -5 & 11 & 1 \\ 0 & 9 & -5 & 11 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

$$\sim -R_2 + R_3 \to R_3 \left[\begin{array}{cccc|c} 2 & -1 & 1 & -3 & 3 \\ 0 & 9 & -5 & 11 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

$$\sim R_3 \leftrightarrow R_4 \left[\begin{array}{cccc|c} 2 & -1 & 1 & -3 & 3 \\ 0 & 9 & -5 & 11 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \begin{array}{l} -R_3 + R_1 \to R_1 \\ 5R_3 + R_2 \to R_2 \end{array} \left[\begin{array}{ccccc} 2 & -1 & 0 & -2 & 2 \\ 0 & 9 & 0 & 6 & 6 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \begin{array}{l} 3R_1 \to R_1 \\ \frac{1}{3}R_2 \to R_2 \end{array} \left[\begin{array}{ccccc} 6 & -3 & 0 & -6 & 6 \\ 0 & 3 & 0 & 2 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim R_2 + R_1 \to R_1 \left[\begin{array}{ccccc} 6 & 0 & 0 & -4 & 8 \\ 0 & 3 & 0 & 2 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \begin{array}{l} \frac{1}{6}R_1 \to R_1 \\ \frac{1}{3}R_2 \to R_2 \end{array} \left[\begin{array}{ccccc} 1 & 0 & 0 & -2/3 & 4/3 \\ 0 & 1 & 0 & 2/3 & 2/3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_4 = t$, $t \in \mathbb{R}$

then
$$\begin{aligned} x_1 &= 4/3 + \tfrac{2}{3}t \\ x_2 &= 2/3 - 2/3\, t \qquad t \in \mathbb{R} \\ x_3 &= 1 + t \\ x_4 &= t \end{aligned}$$

b) $t=0$ $(x_1, x_2, x_3, x_4) = (4/3, 2/3, 1, 0)$

$t=1$ $(x_1, x_2, x_3, x_4) = (2, 0, 2, 1)$

Question 2. Consider the matrices:

$$A=\begin{bmatrix}1&-1&2\\0&2&-3\\2&3&4\end{bmatrix},\ B=\begin{bmatrix}2&2&1\\0&-4&0\end{bmatrix}\ C=\begin{bmatrix}0&1\\-3&0\\3&-1\end{bmatrix}\ D=\begin{bmatrix}0&1\\1&-2\end{bmatrix}\quad D^t=\begin{bmatrix}0&1\\1&-2\end{bmatrix}\quad (D^t)^{-1}=\begin{bmatrix}2&1\\1&0\end{bmatrix}$$

a. *(1 mark)* Compute the following, if possible.

$C-D$ undefined since dimension are different

b. *(2 marks)* Compute the following, if possible.

$(BC)^t$

$$BC=\begin{bmatrix}2&2&1\\0&-4&0\end{bmatrix}\begin{bmatrix}0&1\\-3&0\\3&-1\end{bmatrix}=\begin{bmatrix}-3&1\\12&0\end{bmatrix}$$

c. *(2 marks)* Compute the following, if possible.

A^2

d. *(2 marks)* Compute the following, if possible.

$\text{tr}(A^2+CB)$

So $(BC)^t=\begin{bmatrix}-3&12\\1&0\end{bmatrix}$ $\quad ((BC)^t)^{-1}=\begin{bmatrix}0&1\\1/12&1/4\end{bmatrix}$

g. *(5 marks)* Find E, if possible.

$(2I-(DE)^t)^{-1}=(BC)^t$

$$A^2=AA=\begin{bmatrix}1&-1&2\\0&2&-3\\2&3&4\end{bmatrix}\begin{bmatrix}1&-1&2\\0&2&-3\\2&3&4\end{bmatrix}=\begin{bmatrix}5&3&13\\-6&-5&-18\\10&16&11\end{bmatrix}$$

$$CB=\begin{bmatrix}0&1\\-3&0\\3&-1\end{bmatrix}\begin{bmatrix}2&2&1\\0&-4&0\end{bmatrix}=\begin{bmatrix}0&-4&0\\-6&-6&-3\\6&10&3\end{bmatrix}$$

So $\text{tr}(A^2+CB)$

$$=\text{tr}\left(\begin{bmatrix}5&-1&13\\-12&-11&-21\\16&26&14\end{bmatrix}\right)$$

$$=5+(-11)+14=8$$

$$(2I-(DE)^t)^{-1}=(BC)^t$$

$$((2I-(DE)^t)^{-1})^{-1}=((BC)^t)^{-1}$$

$$2I-(DE)^t=((BC)^t)^{-1}$$

$$E^tD^t=2I-((BC)^t)^{-1}$$

$$E^t=(2I-((BC)^t)^{-1})(D^t)^{-1}$$

$$E^t=\left(2\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}0&1\\1/12&1/4\end{bmatrix}\right)\begin{bmatrix}2&1\\1&0\end{bmatrix}$$

$$E^t=\begin{bmatrix}2&-1\\-1/12&7/4\end{bmatrix}\begin{bmatrix}2&1\\1&0\end{bmatrix}$$

$$E^t=\begin{bmatrix}3&2\\\frac{19}{12}&-1/12\end{bmatrix}\qquad E=\begin{bmatrix}3&19/12\\2&-1/12\end{bmatrix}$$

Question 3. *(4 marks)* Given the following augmented matrix in row-echelon form, solve the system using back substitution.

$$\left[\begin{array}{ccccc|c} 1 & 5 & -3 & 2 & 4 & 0 \\ 0 & 0 & 1 & -2 & 3 & -1 \\ 0 & 0 & 0 & 1 & 2 & 2 \end{array}\right]$$

$x_2 = s \quad s \in \mathbb{R}$

$x_5 = t \quad , t \in \mathbb{R}$

(1) $x_1 + 5x_2 - 3x_3 + 2x_4 + 4x_5 = 0$

(2) $x_3 - 2x_4 + 3x_5 = -1$

(3) $x_4 + 2x_5 = 2$

sub x_5 into (3)

$x_4 + 2t = 2$

$x_4 = 2 - 2t$

sub x_4, x_5 into (2)

$x_3 - 2(2-2t) + 3t = -1$

$x_3 = 3 - 7t$

sub x_2, x_3, x_4, x_5 into

$x_1 + 5s - 3(3-7t) + 2(2-2t) + 4t = 0$

$x_1 = -5s + 5 - 21t$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -5s+5-21t \\ s \\ 3-7t \\ 2-2t \\ t \end{pmatrix}$$

Question 4.[1] Let A and B be invertible $n \times n$ matrices, and AB is its own inverse (i.e. $(AB)^{-1} = AB$)

a. *(2 marks)* Prove that BA is invertible and is its own inverse.

b. *(2 marks)* Evaluate and simplify $(AB+I)^2$

c. *(1 marks)* Evaluate and simplify $(AB+I)^8$

a)

b) $(AB+I)^2 = (AB+I)(AB+I)$

$= ABAB + IAB + ABI + II$

$= I + AB + AB + I$

$= 2I + 2AB$

$= 2(I + AB)$

c) $(AB+I)^8 = ((AB+I)^2)^4$

$= (2(I+AB))^4$

$= 2^4((I+AB)^2)^2$

$= 2^4(2(I+AB))^2$

$= 2^4 2^2 (I+AB)^2$

$= 2^4 2^2 2(I+AB) = 2^7(I+AB)$

a) From premise

$ABAB = I$

$BABAB = BI$

$BABABB^{-1} = BB^{-1}$ from premise

$BABA = I$

$\therefore BA$ is invertible and is its own inverse.

[1] From a John Abbot Final Examination

Question 5. Let A and B be $n \times n$ matrices. Prove that

a. *(2 marks)* if $AB = BA$ then $A^tB^t = B^tA^t$

b. *(2 marks)* if $A^tB^t = B^tA^t$ then $AB = BA$

a) $A^tB^t = (BA)^t \overset{\text{using the premise } AB=BA}{=} (AB)^t = B^tA^t$

b) $AB = ((AB)^t)^t = (B^tA^t)^t \underbrace{=}_{\text{Using the premise } A^tB^t = B^tA^t} (A^tB^t)^t = (B^t)^t(A^t)^t = BA$

Question 6.[2] Let

$$A = \begin{bmatrix} 2 & 1 \\ x & y \end{bmatrix}$$

a. *(2 marks)* Find x and y such that $A^2 = 0$, if possible.

b. *(2 marks)* Find x and y such that $A^2 = I$, if possible.

a) $A^2 = \begin{bmatrix} 2 & 1 \\ x & y \end{bmatrix}\begin{bmatrix} 2 & 1 \\ x & y \end{bmatrix} = \begin{bmatrix} 4+x & 2+y \\ 2x+xy & x+y^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$4 + x = 0$ $\quad\quad$ $2+y = 0$

$x = -4$ $\quad\quad$ $y = -2$

verify other equations

$2(-4) + (-4)(-2)$ $\quad\quad$ $-4 + (-2)^2 = 0$

$= 0$

b) $\begin{bmatrix} 4+x & 2+y \\ 2x+xy & x+y^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad$ $4+x = 1$ $\quad$ $2+y = 0$

$x = -3$ $\quad\quad$ $y = -2$

verify other equations

$2(-3) + (-3)(-2) = 0$

and

$-3 + (-2)^2 = 1$ ✓

[2]From a John Abott Final Examination

Question 7. Consider the following system:

$$\begin{array}{rcrcrcl} x & + & y & + & 7z & = & -1 \\ x & + & 2y & + & 3z & = & 3 \\ 2x & + & 3y & + & a(a^2+11)z & = & 3a+5 \end{array}$$

where $a \in \mathbb{R}$, determine the values of a so that the system has

a. *(2 marks)* a unique solution, justify.

b. *(2 marks)* infitely many solutions, justify.

c. *(2 marks)* no solutions, justify.

$$\left[\begin{array}{ccc|c} 1 & 1 & 7 & -1 \\ 1 & 2 & 3 & 3 \\ 2 & 3 & a^2+11 & 3a+5 \end{array}\right]$$

$$\begin{array}{r} \sim -R_1+R_2 \to R_2 \\ -2R_1+R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 7 & -1 \\ 0 & 1 & -4 & 4 \\ 0 & 1 & a^2-3 & 3a+7 \end{array}\right]$$

$$\sim -R_2+R_3 \to R_3 \left[\begin{array}{ccc|c} 1 & 1 & 7 & -1 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & a^2+1 & 3a+3 \end{array}\right]$$

a) unique solution if # var = # leading 1 : with leading 1 in var. columns.

So $a^2+1 \neq 0$, no real makes $a^2=-1$

$\therefore \forall a \in \mathbb{R}$ the sys. has a unique sol.

b) infinitely many solutions if # var > # leading 1.

In this case

$a^2-1=0$ & $3a+3=0$

Impossible since no real number makes $a^2=-1$

c) no solutions if a contradition of the form

$0x+0y+0z=b$ where $b \neq 0$

In this case

$a^2-1=0$ and $3a+3 \neq 0$

Impossible since no real number makes $a^2=-1$

Bonus Question. *(5 marks)* Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A, B, C, and D are all $n \times n$ and each commutes with all the others. If $M^2 = 0$, show $(A+D)^3 = 0$.

$$(A+D)(A+D)(A+D) = (A+D)(A^2 + AD + DA + D^2)$$
$$= (A+D)(A^2 + AD + AD + D^2) \quad \text{from } ①$$
$$= (A+D)(-BC + AD + AD - BC)$$
$$= 2(A+D)(AD - BC)$$
$$= 2(AAD - ABC + DAD - BCD)$$
$$= 2(A^2D - ABC + ADD - BCD)$$
$$= 2(-BCD - ABC + AD^2 - BCD)$$
$$= 2(-BCD - ABC + A(-BC) - BCD)$$
$$= 2(-CDB - CAB - CAB - CDB)$$

$$M^2 = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A^2 + BC & AB + BD \\ CA + DC & CB + D^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So $A^2 + BC = 0$ $\qquad$ $CB + D^2 = 0$

① $A^2 = -BC$ $\qquad$ ② $D^2 = -CB = -BC$

$\therefore A^2 = D^2$

and $\quad CA + DC = 0$ $\qquad$ and $\quad AB + BD = 0$

$CA + CD = 0$ $\qquad$ $AB + DB = 0$

③ $C(A+D) = 0$ $\qquad$ $(A+D)B = 0$

$$= 2(-2CDB - 2CAB)$$
$$= 2(-2)C(D+A)B$$
$$= -4\, C(A+D)B$$
$$= -4\, 0\, B \quad \text{from } ③$$
$$= 0$$