Dawson College: Linear Algebra: 201-NYC-05-S05: Fall 2014

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Test 2

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Given

$$A=\begin{bmatrix}9&1&1&1&9&3&4&1&0&9\\9&9&1&1&4&9&2&7&7&9\\0&0&7&1&1&4&9&2&7&7\\0&0&0&7&1&1&4&9&2&7\\0&0&0&0&2&1&1&4&9&2\\0&0&0&0&0&1&1&1&4&9\\0&0&0&0&0&0&4&1&1&4\\0&0&0&0&0&0&0&1&1&1\\0&0&0&0&0&0&0&0&1&1\\0&0&0&0&0&0&0&0&1&2\end{bmatrix},\ B=\begin{bmatrix}-2&3&2&6\\1&2&3&-3\\-3&0&1&9\\4&5&5&-12\end{bmatrix},\ C=\begin{bmatrix}1&2&3\\3&1&0\\3&2&1\end{bmatrix},\ D=\begin{bmatrix}-2&3&2&3\\1&0&3&0\\-3&0&1&2\\4&5&0&-1\end{bmatrix}$$

a. *(3 marks)* Is A invertible, justify.

b. *(2 marks)* Is B invertible, justify.

c. *(2 marks)* Compute the determinant of C.

d. *(3 marks)* Compute the determinant of D.

a) $|A| = \begin{matrix}-R_1+R_2\to R_2\\ -R_9+R_{10}\to R_{10}\end{matrix}$ $|F| = 9(8)(7)(7)(2)(1)(4)(1)(1)(1) \neq 0$ $\therefore$ A is invertible

b) $\det B = 0$ since $-3C_1 = C_4$
$\therefore$ B is not invertible.

c) $\begin{vmatrix}1&2&3\\3&1&0\\3&2&1\end{vmatrix}\begin{matrix}1&2\\3&1\\3&2\end{matrix} = 1(1)(1)+2(0)(3)+3(3)(2)-3(1)(3)-1(0)(2)-2(3)(1)$
$= 1+18-9-6 = 4$

d) $= d_{12}C_{12} + d_{22}C_{22} + d_{32}C_{32} + d_{42}C_{42}$

$= 3(-1)\begin{vmatrix}1&3&0\\-3&1&2\\4&0&-1\end{vmatrix} + 0 + 0 + 5\begin{vmatrix}-2&2&3\\1&3&0\\-3&1&2\end{vmatrix}$

$= -3\left[4\begin{vmatrix}3&0\\1&2\end{vmatrix} + (-1)\begin{vmatrix}1&3\\-3&1\end{vmatrix}\right] + 5\left[(-1)\begin{vmatrix}2&3\\1&2\end{vmatrix} + 3\begin{vmatrix}-2&3\\-3&2\end{vmatrix}\right]$

$= -3\left[4(6-0) - (1+9)\right] + 5\left[-(4-3) + 3(4+9)\right]$

$= -3[14] + 5[+14]$

$= -42 + 70 = 28$

$= -92$

Question 2. Consider

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & -2 \\ 0 & 0 & 3 \end{bmatrix}.$$

a. *(4 marks)* Find A^{-1} using the adjoint of A.

b. *(4 marks)* Evaluate $\det(\det(A)\,\mathrm{adj}(A^{-1}))$, justify fully.

c. *(2 marks)* Using a. solve $Ax = b$ where $x = [x_1, x_2, x_3]'$ and $b = [1, -2, 1]'$.

a) $A^{-1} = \frac{1}{\det A}\,\mathrm{adj}\,A$

$\det A = \overbrace{a_{31}}^{0} C_{31} + \overbrace{a_{32}}^{0} C_{32} + a_{33} C_{33}$

$= 3(-1)^{3+3}\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 3[1+4] = 15$

$C_{11} = \begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix} = 3$, $C_{12} = (-1)\begin{vmatrix} 2 & -2 \\ 0 & 3 \end{vmatrix} = -6$, $C_{13} = \begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix} = 0$, $C_{21} = (-1)\begin{vmatrix} -2 & 3 \\ 0 & 3 \end{vmatrix} = 6$

$C_{22} = \begin{vmatrix} 1 & 3 \\ 0 & 3 \end{vmatrix} = 3$, $C_{23} = (-1)\begin{vmatrix} 1 & -2 \\ 0 & 0 \end{vmatrix} = 0$, $C_{31} = \begin{vmatrix} -2 & 3 \\ 1 & -2 \end{vmatrix} = 1$

$C_{32} = (-1)\begin{vmatrix} 1 & 3 \\ 2 & -2 \end{vmatrix} = +8$ $\quad C_{33} = \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 5$

$\mathrm{adj}\,A = [\text{matrix of cofactor}]^T = \begin{bmatrix} 3 & -6 & 0 \\ 6 & 3 & 0 \\ \cancel{8}\,1 & 8 & 5 \end{bmatrix}^T = \begin{bmatrix} 3 & 6 & 1 \\ -6 & 3 & 8 \\ 0 & 0 & 5 \end{bmatrix}$

$A^{-1} = \frac{1}{15}\begin{bmatrix} 3 & 6 & 1 \\ -6 & 3 & 8 \\ 0 & 0 & 5 \end{bmatrix}$

c) $x = A^{-1}b$

$x = \frac{1}{15}\begin{bmatrix} 3 & 6 & 1 \\ -6 & 3 & 8 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} -8 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} -8/15 \\ -4/15 \\ 5/15 \end{bmatrix}$

b) $\det(A)^3 \det(\mathrm{adj}(A^{-1}))$ since $\det(A)$ is a constant

$= (15)^3 \det(A^{-1})^{3-1}$ from thm seen in class

$= 15^3 \left(\frac{1}{\det A}\right)^2$ since A is invertible

$= 15^3 \frac{1}{15^2} = 15$

Question 3. *(3 marks)* Solve *only* for x_3 using Cramer's rule.

$$\begin{array}{rcrcrcrcrcr} 3x_1 & - & x_2 & + & x_3 & - & 3x_4 & + & 4x_5 & = & 2 \\ & & 3x_2 & & & & & - & 10x_5 & = & 3 \\ & & & - & x_3 & - & 2x_4 & + & 3x_5 & = & -3 \\ & & & & & - & x_4 & - & 3x_5 & = & 0 \\ & & & & & & & & 4x_5 & = & 0 \end{array}$$

$$A_3 = \begin{bmatrix} 3 & -1 & 2 & -3 & 4 \\ 0 & 3 & 3 & 0 & -10 \\ 0 & 0 & -3 & -2 & 3 \\ 0 & 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

$Ax = b$

where $A = \begin{bmatrix} 3 & -1 & 1 & -3 & 4 \\ 0 & 3 & 0 & 0 & -10 \\ 0 & 0 & -1 & -2 & 3 \\ 0 & 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

$b = \begin{bmatrix} 2 \\ 3 \\ -3 \\ 0 \\ 0 \end{bmatrix}$

$$X_3 = \frac{\det A_3}{\det A} = \frac{\cancel{3}(\cancel{3})(-3)(\cancel{-1})(\cancel{4})}{\cancel{3}(\cancel{3})(-1)(\cancel{-1})(\cancel{4})} = 3$$

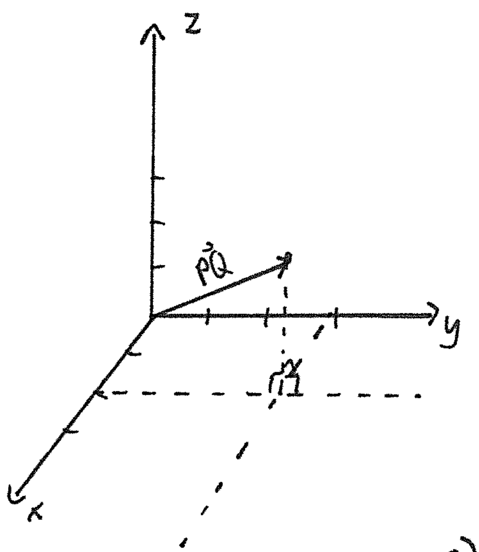
Question 4. Given $P(0,2,4)$ and $Q(2,5,7)$ $\quad \vec{PQ} = Q - P = (2,5,7) - (0,2,4) = (2,3,3)$

a. *(2 marks)* Sketch the vector $\vec{PQ}$ with the initial point located at the origin.

b. *(3 marks)* Find the radian measure of the angle θ (with $0 \le \theta \le \pi$) between $\vec{u} = (2,-1,0)$ and $\vec{PQ}$.

c. *(2 marks)* Prove or Disprove: The linear combinations $a_1\vec{v}_1 + a_2\vec{v}_2$ and $b_1\vec{v}_1 + b_2\vec{v}_2$ can only be equal if $a_1 = b_1$ and $a_2 = b_2$

a)



b) $\vec{u} \cdot \vec{PQ} = \|\vec{u}\| \|\vec{PQ}\| \cos\theta$

$(2,-1,0)\cdot(2,3,3) = \|(2,-1,0)\| \|(2,3,3)\| \cos\theta$

$2(2) + (-1)(3) + 0(3) = \sqrt{(2)^2+(-1)^2+0^2}\sqrt{2^2+3^2+3^2}\cos\theta$

$1 = \sqrt{5}\sqrt{22}\cos\theta$

$\theta = \arccos\left(\frac{1}{\sqrt{5}\sqrt{22}}\right)$

$\theta =$

c) Disprove: If $\vec{v}_2 = k\vec{v}_1$ then

$a_1\vec{v}_1 + a_2 k\vec{v}_1 = b_1\vec{v}_1 + b_2 k\vec{v}_1$

$(a_1 + a_2 k)\vec{v}_1 = (b_1 + k b_2)\vec{v}_1$

iff $\quad a_1 + a_2 k = b_1 + k b_2$

∞ solutions.

Question 5. *(2 marks)* Let A and B be $n \times n$ matrices, $AB = -BA$, and n is odd, show that either A or B has no inverse.

$$AB = -BA$$
$$\det(AB) = \det(-BA)$$
$$\det(AB) = (-1)^n \det(BA)$$
$$\det A \det B = -\det B \det A$$

where $(-1)^n = -1$ since n is odd.

Since $K = -K$ iff $K = 0$

$\therefore$ Either $\det A = 0$ or $\det B = 0$

$\therefore$ Either A or B is singular.

Question 6. *(2 marks)* Show that $\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2(\|\vec{u}\|^2 + \|\vec{v}\|^2)$

$$\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2$$
$$= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) + (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$
$$= \vec{u}\cdot\vec{u} + \cancel{\vec{u}\vec{v}} + \cancel{\vec{v}\vec{u}} + \vec{v}\cdot\vec{v} + \vec{u}\cdot\vec{u} - \cancel{\vec{u}\vec{v}} - \cancel{\vec{v}\cdot\vec{u}} + \vec{v}\cdot\vec{v}$$
$$= 2\vec{u}\cdot\vec{u} + 2\vec{v}\cdot\vec{v}$$
$$= 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$$
$$= 2(\|\vec{u}\|^2 + \|\vec{v}\|^2)$$

Question 7. *(2 marks)* Prove: If B and C are $n \times n$ matrices, $A = B^T C + C^T B$ is invertible then A^{-1} is symmetric.

$$(A^{-1})^T$$
$$= (A^T)^{-1}$$
$$= ((B^TC + C^TB)^T)^{-1}$$
$$= ((B^TC)^T + (C^TB)^T)^{-1}$$
$$= (C^T(B^T)^T + B^T(C^T)^T)^{-1}$$
$$= (C^TB + B^TC)^{-1}$$
$$= (B^TC + C^TB)^{-1} = A^{-1}$$

$\therefore A^{-1}$ is symmetric

Question 8. *(4 marks)* Write the given matrix as a product of elementary matrices

$$A=\begin{bmatrix}0&0&1\\1&3&0\\2&4&0\end{bmatrix} \sim R_1 \leftrightarrow R_3 \begin{bmatrix}2&4&0\\1&3&0\\0&0&1\end{bmatrix} \sim \tfrac{1}{2}R_1 \to R_1 \begin{bmatrix}1&2&0\\1&3&0\\0&0&1\end{bmatrix} \sim -R_1+R_2 \to R_2 \begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\sim -2R_2+R_1 \to R_1 \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

So

$$\underbrace{E_4E_3E_2E_1}_{A^{-1}}A = I$$

$$A^{-1} = E_4E_3E_2E_1$$

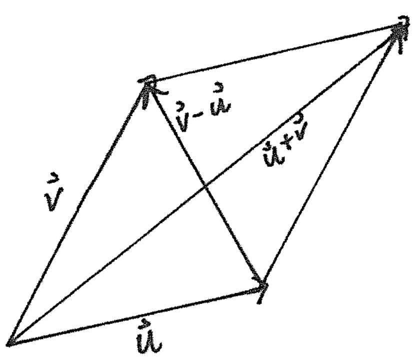
$$(A^{-1})^{-1} = (E_4E_3E_2E_1)^{-1}$$

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$$

where $E_1^{-1} = \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$ $\quad E_3^{-1} = \begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix}$

$E_2^{-1} = \begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}$ $\quad E_4^{-1} = \begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}$

Bonus Question. *(4 marks)* Show that if the diagonals of a parallelogram are perpendicular, it is necessarily a rhombus.



$$(\vec{u}+\vec{v})\cdot(\vec{v}-\vec{u})$$
$$= \vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - \vec{u}\cdot\vec{v}$$
$$= \vec{v}\cdot\vec{v} - \vec{u}\cdot\vec{u}$$
$$= \|\vec{v}\|^2 - \|\vec{u}\|^2$$

If diagonals are perpendicular

$$(\vec{u}+\vec{v})\cdot(\vec{v}-\vec{u}) = 0$$
$$\|\vec{v}\|^2 - \|\vec{u}\|^2 = 0$$
$$\|\vec{v}\|^2 = \|\vec{u}\|^2$$
$$\|\vec{v}\| = \|\vec{u}\|$$

$\therefore$ the parallelogram is a rhombus.