Dawson College: Linear Algebra: 201-NYC-05-S05: Fall 2014

Name: Y. Lamontagne
Student ID: ____________________

Test 3

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Given

$A(1,2)$, $B(-1,1)$, $C(3,1)$

a. *(3 marks)* Find the area of the parallelogram generated by $\vec{AB}$ and $\vec{AC}$.

b. *(1 mark)* Find the area of the triangle with vertices A, B and C.

c. *(2 marks)* Find the length of the altitude of the triangle ABC from vertex C to side AB.

d. *(1 mark)* Find the shortest distance from C to the line that contains A and B.

a. Lets embed A, B, C on the xy plane of $\mathbb{R}^3$ i.e.

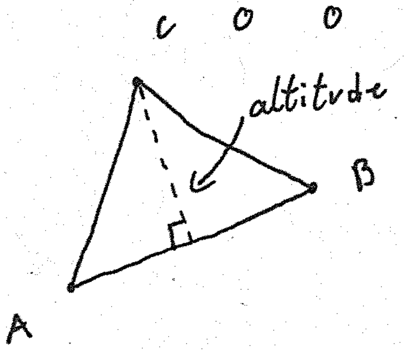
$A(1,2,0)$ $B(-1,1,0)$, $C(3,1,0)$

$\vec{AB} = B - A = (-1,1,0) - (1,2,0) = (-2,-1,0)$

$\vec{AC} = C - A = (3,1,0) - (1,2,0) = (2,-1,0)$

$$\text{Area} = \|\underset{\substack{-2\\-1\\0}}{\vec{AB}} \times \underset{\substack{2\\-1\\0}}{\vec{AC}}\| = \left\| \left(\begin{vmatrix} -1 & -1 \\ 0 & 0 \end{vmatrix}, -\begin{vmatrix} -2 & 2 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} -2 & 2 \\ -1 & -1 \end{vmatrix} \right) \right\| = \|(0,0,4)\| = 4$$

b)



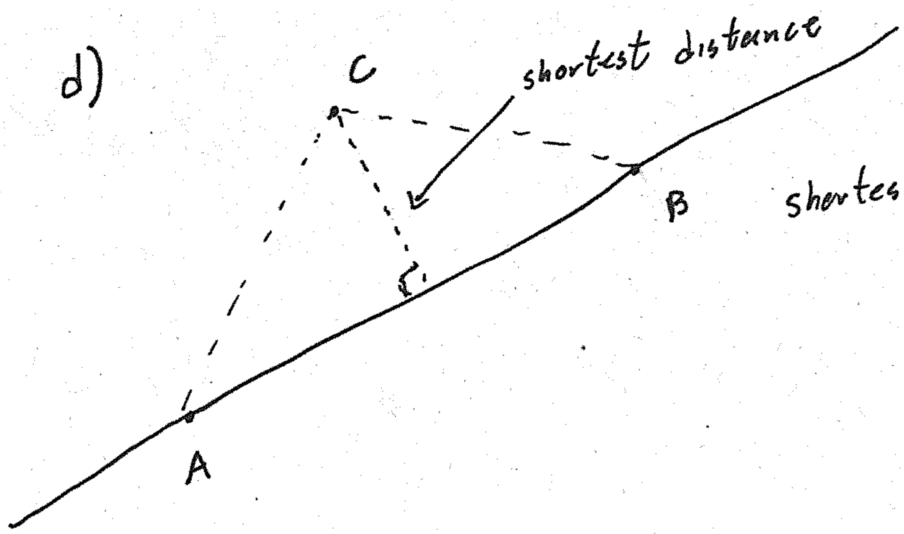
$$\text{Area} = \frac{\|\vec{AB} \times \vec{AC}\|}{2} = \frac{4}{2} = 2$$

c) $$\text{Area} = \frac{\text{base (height)}}{2}$$

$$2 = \frac{\text{base (altitude)}}{2}$$

$$\text{altitude} = \frac{4}{\text{base}} = \frac{4}{\|\vec{AB}\|} = \frac{4}{\sqrt{(-2)^2+(-1)^2+0^2}} = \frac{4}{\sqrt{5}}$$

d)



shortest distance = altitude of triangle ABC

$$= \frac{4}{\sqrt{5}}$$

Question 2. Given

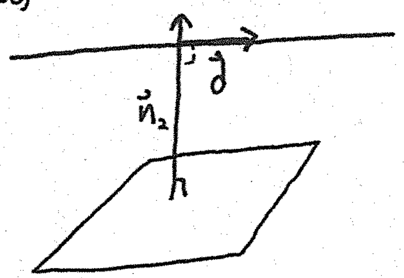
$$
\begin{array}{lrcll}
\mathscr{L}_1: & (x,y,z) & = & (2+5t, 1+t, -t) & t \in \mathbb{R} \\
\mathscr{L}_2: & (x,y,z) & = & (7+2t, 4, 10t) & t \in \mathbb{R} \\
\mathscr{L}_3: & (x,y,z) & = & (9-t, 2, 9-5t) & t \in \mathbb{R} \\
\mathscr{P}_1: & x-2y+3z-11 & = & 0 & \\
\mathscr{P}_2: & -5x-y+z+31 & = & 0 &
\end{array}
$$

a. *(2 marks)* Are $\mathscr{P}_2$ and $\mathscr{L}_3$ parallel, perpendicular, or neither, justify?

b. *(4 marks)* Are $\mathscr{L}_2$ and $\mathscr{P}_1$ parallel, perpendicular, or neither, justify? If they intersect find the point of intersection.

c. *(5 marks)* Are $\mathscr{L}_1$ and $\mathscr{L}_2$ parallel, perpendicular, or neither, justify? Find the shortest distance between the two lines using projections.

a)



$\vec{d}_3 = (-1, 0, -5)$

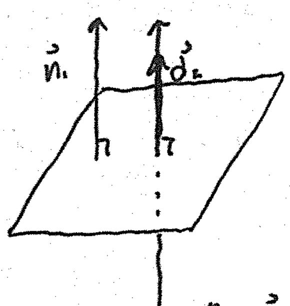
$\vec{n}_2 = (-5, -1, 1)$

$\vec{d}_3 \cdot \vec{n}_2 = (-1)(-5) + (0)(-1) + (-5)(1) = 0$

$\therefore$ the line is parallel to the plane.

b) $\vec{n}_1 = (1, -2, 3)$ $\quad \vec{d}_2 = (2, 0, 10)$ $\quad \vec{n}_1 \cdot \vec{d}_2 = (1,-2,3)\cdot(2,0,10) \neq 0$ $\quad \therefore$ not parallel

$\vec{n}_1 \neq k\vec{d}_2$ $\quad \therefore$ not perpendicular

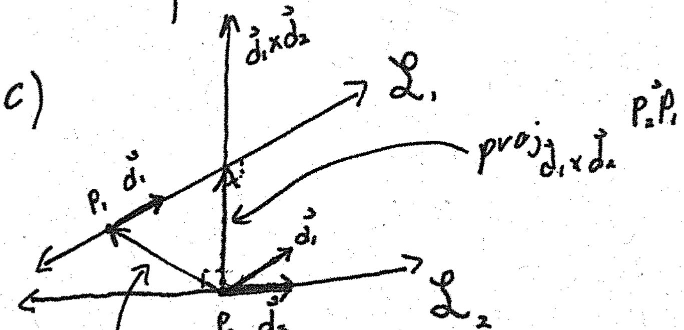


$x = 7+2t$
$y = 4$
$z = 10t$

$$
\begin{aligned}
x - 2y + 3z &= 11 \\
(7+2t) - 2(4) + 3(10t) &= 11 \\
7 + 2t - 8 + 30t &= 11 \\
32t &= 12 \\
t = \tfrac{12}{32} &= \tfrac{3}{8}
\end{aligned}
$$

$$
\begin{aligned}
(x,y,z) &= \left(7 + 2\left(\tfrac{3}{8}\right), 4, 10\left(\tfrac{3}{8}\right)\right) \\
&= \left(\tfrac{31}{4}, 4, \tfrac{15}{4}\right)
\end{aligned}
$$

c)



$$
\begin{aligned}
\vec{P_2P_1} &= P_1 - P_2 \\
&= (2, 1, 0) - (7, 4, 0) \\
&= (-5, -3, 0)
\end{aligned}
$$

$$
\begin{aligned}
&\text{Proj}_{\vec{d}_1 \times \vec{d}_2} \vec{P_2P_1} \\
&= \frac{\vec{P_2P_1} \cdot (\vec{d}_1 \times \vec{d}_2)}{(\vec{d}_1 \times \vec{d}_2)\cdot(\vec{d}_1 \times \vec{d}_2)} \, \vec{d}_1 \times \vec{d}_2 \\
&= \frac{10(-5) + (-52)(-3) + (-2)(0)}{10(10) + (-52)(-52) + (-2)(-2)} (10, -52, -2) \\
&= \frac{106}{2804} (10, -52, -2) \\
&= \frac{53}{702} (5, -26, -1)
\end{aligned}
$$

$$
\vec{d}_1 \times \vec{d}_2 = \left(\begin{vmatrix} 1 & 0 \\ -1 & 10 \end{vmatrix}, - \begin{vmatrix} 5 & 2 \\ -1 & 10 \end{vmatrix}, \begin{vmatrix} 5 & 2 \\ 1 & 0 \end{vmatrix} \right)
$$

$\begin{matrix} 5 & 2 \\ 1 & 0 \\ -1 & 10 \end{matrix} \quad = (10, -52, -2)$

$\vec{d}_1 \cdot \vec{d}_2 = 0$, so their directions are perpendicular

$$
\begin{aligned}
\text{distance} &= \|\text{Proj}_{\vec{d}_1 \times \vec{d}_2} \vec{P_2P_1}\| \\
&= \left\| \frac{53}{702}(5, -26, -1) \right\| = \frac{53}{702}\sqrt{5^2 + (-26)^2 + (-1)^2} \\
&= \frac{53}{702}\sqrt{702} = \frac{53}{\sqrt{702}}
\end{aligned}
$$

Question 4. Let $\mathscr{H} = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ with the usual addition and scalar multiplication.

a. *(2 marks)* Give an example of a non-zero matrix in $\mathscr{H}$? Justify.

b. *(2 marks)* Does $\mathscr{H}$ satisfy closure under vector addition? Justify.

c. *(2 marks)* Does $\mathscr{H}$ contain the zero vector of $\mathscr{M}_{2\times 2}$ (the vector space of 2×2 matrices)? Justify.

d. *(2 marks)* Does $\mathscr{H}$ satisfy closure under scalar multiplication? Justify.

e. *(2 marks)* Is $\mathscr{H}$ a vector subspace of $\mathscr{M}_{2\times 2}$ (the vector space of 2×2 matrices)? Justify.

a) $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \in H$ since $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

b) Let $A, B \in H$ then $A + B \in H$ since $(A+B)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} + B\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ since $A, B \in H$.

c) $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in H$ since $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

d) Let $A \in H$ and $k \in \mathbb{R}$ then $kA \in H$ since $(kA)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = k\left(A\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = k\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ since $A \in H$

$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

e) H is a subspace since it is closed under scalar multiplication and vector addition.

Question 5. *(2 marks)* Prove or disprove: For all vector vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in 3-space, the vectors $(\vec{u} \times \vec{v}) \times \vec{w}$ and $\vec{u} \times (\vec{v} \times \vec{w})$ are the same.

disprove: $\vec{u} = (1,1,1)$
$\vec{v} = (1,0,0)$
$\vec{w} = (0,1,0)$

$\vec{u} \times \vec{v} = \left(\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \right) = (0, 1, -1)$

$(\vec{u} \times \vec{v}) \times \vec{w} = \left(\begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix}, -\begin{vmatrix} 0 & 0 \\ -1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} \right) = (1, 0, 0)$

$\vec{v} \times \vec{w} = (0,0,1)$

$\vec{u} \times (\vec{v} \times \vec{w}) = \left(\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} \right) = (1, -1, 0)$

Question 6. *(2 marks)* If $\vec{a}$ and $\vec{b}$ are orthogonal vectors, then for every nonzero vector $\vec{u}$, we have

$\text{proj}_{\vec{a}}(\text{proj}_{\vec{b}}(\vec{u})) = \vec{0}$

$$\text{proj}_{\vec{a}}(\text{proj}_{\vec{b}}\,\vec{u}) = \text{proj}_{\vec{a}}\left(\frac{\vec{u}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\,\vec{b}\right)$$

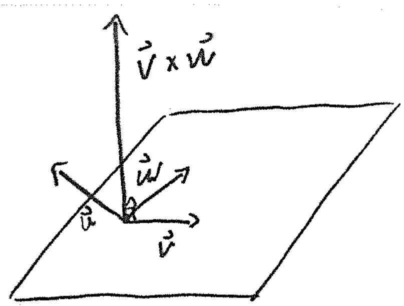
$$= \frac{\vec{a}\cdot\left(\frac{\vec{u}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\,\vec{b}\right)}{\vec{a}\cdot\vec{a}}\,\vec{a}$$

$$= \left(\frac{\vec{u}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)\frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\,\vec{a}$$

$$= \left(\frac{\vec{u}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)\frac{0}{\vec{a}\cdot\vec{a}}\,\vec{a} \qquad \text{since } \vec{a}\perp\vec{b}$$

$$= \vec{0}$$

Question 7. *(2 marks)* Show that if $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^3$ no which are collinear, then $\vec{u}\times(\vec{v}\times\vec{w})$ lies in the plane determined by $\vec{v}$ and $\vec{w}$.



Then $\vec{u}\times(\vec{v}\times\vec{w})$ lies on the plane

if $\vec{u}\times(\vec{v}\times\vec{w})\cdot(\vec{v}\times\vec{w}) = 0$

Let $\vec{y} = (y_1, y_2, y_3) = \vec{v}\times\vec{w}$

then $(\vec{u}\times\vec{y})\cdot\vec{y} = \vec{y}\cdot(\vec{u}\times\vec{y})$

$$= \begin{vmatrix} y_1 & y_2 & y_3 \\ u_1 & u_2 & u_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = 0$$

Question 8. *(2 marks)* Prove: If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in a vector space V and $\vec{u}+\vec{w} = \vec{v}+\vec{w}$, then $\vec{u} = \vec{v}$. (*cancellation law for vector addition*). **Show EVERY step and justify EVERY step with axiom names when an axiom is used**

$\vec{u}+\vec{w} = \vec{v}+\vec{w}$ — let $\vec{z}$ be the additive inverse of $\vec{w}$,

$(\vec{u}+\vec{w})+\vec{z} = (\vec{v}+\vec{w})+\vec{z}$ — the inverse exists by axiom 5

$\vec{u}+\vec{0} = \vec{v}+\vec{0}$ — by 5

$\vec{u} = \vec{v}$ — by 4

$\vec{u}+(\vec{w}+\vec{z}) = \vec{v}+(\vec{w}+\vec{z})$ by 3

Question 9. *(2 marks)* Determine wether the following is a vector space:

$$\mathscr{P} = \{a_0 + a_1x + a_2x^2 \mid a_0 + a_1 + a_2 = 1\}$$

with the usual polynomial addition and scalar multiplication. i.e.

$$p_1(x) + p_2(x) = (a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$

and

$$kp_1(x) = k(a_0 + a_1x + a_2x^2) = ka_0 + ka_1x + ka_2x^2$$

$\mathscr{P}$ is not a vector space since the $\vec{0} \notin \mathscr{P}$. Let $p(x) = a_0 + a_1x + a_2x^2 \in \mathscr{P}$ and $0(x) = b_0 + b_1x + b_2x^2$

$$0(x) + p(x) = p(x)$$

$$(b_0 + b_1x + b_2x^2) + (a_0 + a_1x + a_2x^2) = a_0 + a_1x + a_2x^2$$

$$(b_0 + a_0) + (b_1 + a_1)x + (b_2 + a_2)x^2 = a_0 + a_1x + a_2x^2$$

$$b_0 + a_0 = a_0 \qquad b_1 + a_1 = a_1 \qquad b_2 + a_2 = a_2$$
$$b_0 = 0 \qquad b_1 = 0 \qquad b_2 = 0$$

So $0(x) \notin \mathscr{P}$ since $b_0 + b_1 + b_2 \neq 1$

Bonus Question. *(5 marks)* If U and W are subspaces of a vector space V, let

$$U \cup W = \{\vec{v} \mid \vec{v} \in U \text{ or } \vec{v} \in W\}.$$

Show that $U \cup W$ is a subspace if and only if $U \subseteq W$ or $W \subseteq U$ (where $A \subseteq B$ means A is a subset of B or is B).

[$\Rightarrow$] Suppose that $U \nsubseteq W$ and $W \nsubseteq U$. Then let $\vec{u} \in U$ s.t. $\vec{u} \notin W$ and let $\vec{w} \in W$ s.t. $\vec{w} \notin U$. It follows that $\vec{u} + \vec{w} \notin U \cup W$ since if it was an element then it would be an element of U or W. But suppose it was an element of U (wolog) then $(u + \vec{w}) - \vec{u}$ would be an element U. So $\vec{w} \in U$ ↯ $\therefore$ $U \subseteq W$ or $W \subseteq U$

[$\Leftarrow$] $U \cup W$ is closed under vector addition and scalar multiplicatio since either $U \subseteq W$ or $W \subseteq U$. $\therefore$ by the subspace test $U \cup W$ is a subspace.