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Quiz 12

This quiz is graded out of 10 marks. No books, calculators, notes or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. §4.3 #17 (5 marks) Prove: The space spanned by two vectors in $\mathbb{R}^3$ is a line through the origin, a plane through the origin, or the origin itself. Let $V = \text{span}(\{\vec{u}_1, \vec{u}_2\})$ then $\vec{v} \in V$, $\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2$ where $c_1, c_2 \in \mathbb{R}$.

Possible cases:

1. $\vec{u}_1 = \vec{u}_2 = \vec{0}$ then $\vec{v} = 0$, the space is the origin
2. $\exists i$ s.t $\vec{u}_i = \vec{0}$ and $\exists j$ s.t $\vec{u}_j \neq \vec{0}$ then $\vec{v} = c_j\vec{u}_j$, the space is a line through the origin
3. $\vec{u}_1 = k\vec{u}_2$ then $\vec{v} = c_1k\vec{u}_2 + c_2\vec{u}_2 = (c_1k + c_2)\vec{u}_2$, the space is a " " " "
4. $\vec{u}_1 \neq k\vec{u}_2$ then $\vec{v} = c_1\vec{u}_1 + c_2 u_2$, the space is a plane through the origin.

Question 2. §4.4 #5 *(5 marks)* Show that the following matrices form a basis for $\mathcal{M}_{2\times 2}$.

$$\underbrace{\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}}_{M_1}, \underbrace{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}}_{M_2}, \underbrace{\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}}_{M_3}, \underbrace{\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}}_{M_4}$$

Lets show that $\{M_1, M_2, M_3, M_4\}$ spans $M_{2\times 2}$. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}$

$M = c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4$

$$\underbrace{\begin{bmatrix} 3 & 0 & 0 & 1 \\ 6 & -1 & -8 & 0 \\ 3 & -1 & -12 & -1 \\ -6 & 0 & -4 & 2 \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

The system is consistent for all a, b, c, d iff $|A| \neq 0$.

$$|A| = 3\begin{vmatrix} -1 & -8 & 0 \\ -1 & -12 & -1 \\ 0 & -4 & 2 \end{vmatrix} - \begin{vmatrix} 6 & -1 & -8 \\ 3 & -1 & -12 \\ -6 & 0 & -4 \end{vmatrix}$$

$$= 3\left[-1\begin{vmatrix} -12 & -1 \\ -4 & 2 \end{vmatrix} + 8\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix}\right] - \left[-6\begin{vmatrix} -1 & -8 \\ -1 & -12 \end{vmatrix} - 4\begin{vmatrix} 6 & -1 \\ 3 & -1 \end{vmatrix}\right]$$

$$= 3\left[-1(-28) - 16\right] - \left[-6(4) - 4(-3)\right] = 48 \neq 0$$

$\therefore \{M_1, M_2, M_3, M_4\}$ spans $M_{2\times 2}$

Lets show that $\{M_1, M_2, M_3, M_4\}$ is linearly independent.

$0 = c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4$

$$\begin{bmatrix} 3 & 0 & 0 & 1 \\ 6 & -1 & -8 & 0 \\ 3 & -1 & -12 & -1 \\ -6 & 0 & -4 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The system has only a trivial solution iff $|A| \neq 0$

$\therefore$ the set is linearly independent

$\therefore$ the set is a basis.