Dawson College: Linear Algebra: 201-NYC-05-S07: Fall 2015

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Test 1

This test is graded out of 46 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1.

a. *(6 marks)* Solve the following system by Gauss-Jordan elimination:

$$\begin{array}{rcrcrcrcl} 3x_1 & + & 3x_2 & + & 7x_3 & - & 3x_4 & = & 0 \\ 2x_1 & + & 3x_2 & + & 3x_3 & + & x_4 & = & 0 \\ 4x_1 & & & + & 17x_3 & - & 2x_4 & = & 0 \\ 9x_1 & + & 6x_2 & + & 27x_3 & - & 4x_4 & = & 0 \end{array}$$

b. *(1 mark)* Find two particular nontrivial solution to the above system.

c. *(1 mark)* Find a solution to the above system where $x_1 = 1$.

$$\begin{bmatrix} 3 & 3 & 7 & -3 & 0 \\ 2 & 3 & 3 & 1 & 0 \\ 4 & 0 & 17 & -2 & 0 \\ 9 & 6 & 27 & -4 & 0 \end{bmatrix}$$

$$\sim \begin{array}{r} \\ 3R_2 \to R_2 \\ 3R_3 \to R_3 \\ \\ \end{array} \begin{bmatrix} 3 & 3 & 7 & -3 & 0 \\ 6 & 9 & 9 & 3 & 0 \\ 12 & 0 & 51 & -6 & 0 \\ 9 & 6 & 27 & -4 & 0 \end{bmatrix}$$

$$\sim \begin{array}{r} \\ -2R_1 + R_2 \to R_2 \\ -4R_1 + R_3 \to R_3 \\ -3R_1 + R_4 \to R_4 \end{array} \begin{bmatrix} 3 & 3 & 7 & -3 & 0 \\ 0 & 3 & -5 & 9 & 0 \\ 0 & -12 & 23 & 6 & 0 \\ 0 & -3 & 6 & 5 & 0 \end{bmatrix}$$

$$\sim \begin{array}{r} \\ \\ 4R_2 + R_3 \to R_3 \\ R_2 + R_4 \to R_4 \end{array} \begin{bmatrix} 3 & 3 & 7 & -3 & 0 \\ 0 & 3 & -5 & 9 & 0 \\ 0 & 0 & 3 & 42 & 0 \\ 0 & 0 & 1 & 14 & 0 \end{bmatrix}$$

$$\sim \begin{array}{r} \\ \\ \frac{1}{3}R_3 \to R_3 \\ -\frac{1}{3}R_3 + R_4 \to R_4 \end{array} \begin{bmatrix} 3 & 3 & 7 & -3 & 0 \\ 0 & 3 & -5 & 9 & 0 \\ 0 & 0 & 1 & 14 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{r} -7R_3 + R_1 \to R_1 \\ \sim 5R_3 + R_2 \to R_2 \\ \\ \\ \end{array} \begin{bmatrix} 3 & 3 & 0 & -101 & 0 \\ 0 & 3 & 0 & 79 & 0 \\ 0 & 0 & 1 & 14 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{array}{r} -R_2 + R_1 \to R_1 \\ \\ \\ \\ \end{array} \begin{bmatrix} 3 & 0 & 0 & -180 & 0 \\ 0 & 3 & 0 & 79 & 0 \\ 0 & 0 & 1 & 14 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{array}{r} \frac{1}{3}R_1 \to R_1 \\ \frac{1}{3}R_2 \to R_2 \\ \\ \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & -60 & 0 \\ 0 & 1 & 0 & 79/3 & 0 \\ 0 & 0 & 1 & 14 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $x_3 = t$ $\quad t \in \mathbb{R}$

$(x_1, x_2, x_3, x_4) = (60t, \frac{-79}{3}t, -14t, t)$

b) if $t = 1$: $(x_1, x_2, x_3, x_4) = (60, \frac{-79}{3}, -14, 1)$

$t = 3$: $(x_1, x_2, x_3, x_4) = (180, -79, -42, 3)$

c) $x_1 = 1$ if $t = 1/60$

$\therefore (x_1, x_2, x_3, x_4) = (1, \frac{-79}{180}, \frac{-14}{60}, \frac{1}{60})$

Question 2. Consider the matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 0 \\ -3 & 4 & 0 \end{bmatrix} C = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} D = \begin{bmatrix} 2 & -2 \\ 1 & -2 \end{bmatrix} F = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 1 & 0 \end{bmatrix}$$

a. *(2 marks)* Compute the following, if possible.

BIF

$$BIF = \begin{bmatrix} 2 & 1 & 0 \\ -3 & 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & -4 \\ 6 & -5 \end{bmatrix}$$

b. *(2 marks)* Compute the following, if possible.

$ACAB$ not defined since $\underbrace{\underset{3\times 3}{A} \quad \underset{2\times 3}{B}}_{\neq}$.

c. *(2 marks)* Compute the following, if possible.

$F(D^{-1})^T B$

$$D^{-1} = \frac{1}{-2}\begin{bmatrix} -2 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} +1 & -1 \\ +\frac{1}{2} & -1 \end{bmatrix}$$

d. *(5 marks)* Determine E, if possible.

$DED^T = \text{tr}(BIF)\ BF$

$$(D^{-1})^T = \begin{bmatrix} 1 & \frac{1}{2} \\ -1 & -1 \end{bmatrix}$$

c)
$$F(D^{-1})^T B = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} +1 & +\frac{1}{2} \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ -3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & +3 & 0 \\ +1 & -5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & +11 & 0 \\ -\frac{1}{2} & +19 & 0 \\ \frac{1}{2} & +3 & 0 \end{bmatrix}$$

d)
$$D^{-1} D E D^T (D^T)^{-1} = D^{-1}(2) BF (D^T)^{-1}$$

$$E = 2\begin{bmatrix} 1 & -1 \\ \frac{1}{2} & -1 \end{bmatrix}\begin{bmatrix} 7 & -4 \\ 6 & -5 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 7 & -4 \\ 6 & -5 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ -5 & 6 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -11 & -1\frac{7}{2} \end{bmatrix}$$

Question 3. Consider

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 9 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

a. *(4 marks)* Find A^{-1}.

b. *(4 marks)* Determine X, if possible.

$$(I + AX)^{-1} = \text{tr(A)}I$$

a) $[A \mid I]$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 4 & 9 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \begin{array}{r} \\ -4R_1 + R_2 \to R_2 \\ -3R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 1 & 0 \\ 0 & -4 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\sim \begin{array}{r} -2R_2 + R_1 \to R_1 \\ \\ 4R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -2 & 0 \\ 0 & 1 & 0 & -4 & 1 & 0 \\ 0 & 0 & 1 & -19 & 4 & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 9 & -2 & 0 \\ -4 & 1 & 0 \\ -19 & 4 & 1 \end{bmatrix}$$

b) $(I + AX)^{-1} = 11I$

$$I + AX = \frac{1}{11}I$$

$$AX = \frac{1}{11}I - I$$

$$AX = \frac{-10}{11}I$$

$$A^{-1}AX = A^{-1}\left(\frac{-10}{11}\right)I$$

$$X = \frac{-10}{11}A^{-1}I$$

$$X = \frac{-10}{11}\begin{bmatrix} 9 & -2 & 0 \\ -4 & 1 & 0 \\ -19 & 4 & 1 \end{bmatrix}$$

Question 4. *(4 marks)* Prove: If AB and BA are both invertible then A and B are both invertible.

Premise:

- AB, BA invertible

Conclusion:

- A, B are invertible

since AB is invertible $AB(AB)^{-1} = I$. So $A^{-1} = B(AB)^{-1}$

since BA is invertible $BA(BA)^{-1} = I$. So $B^{-1} = A(BA)^{-1}$.

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Question 5. *(4 marks)* A square matrix A is called *idempotent* if $A^2 = A$. Prove: If A is idempotent then $A + AB - ABA$ is idempotent for any square matrix B with the same dimension as A.

Premise:

- A is idempotent, $A^2 = A$

Conclusion:

- $A + AB - ABA$ is idempotent

$$
\begin{aligned}
(A + AB - ABA)^2 &= (A + AB - ABA)(A + AB - ABA) \\
&= A^2 + A^2B - A^2BA + ABA + ABAB - ABABA - ABA^2 \\
&\quad - ABA^2B + ABA^2BA \\
&= A + AB - A\cancel{BA} + \cancel{AB}A + \cancel{ABAB} - \cancel{ABAB}A - ABA \\
&\quad - \cancel{ABAB} + \cancel{ABABA} \\
&= A + AB - ABA
\end{aligned}
$$

$\therefore$ $A + AB - ABA$ is idempotent

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Question 6. Let

$$A=\begin{bmatrix}1&-1&3&2\\0&1&4&1\\1&0&8&6\end{bmatrix},\ B=\begin{bmatrix}1&-1&4&5\\-2&1&-11&-8\\-1&2&2&2\end{bmatrix}$$

a. *(5 marks)* Show that A and B have the same reduced row echelon matrix.

b. *(3 marks)* Justify using a. that A and B are row equivalent.

a)

$$A=\begin{bmatrix}1&-1&3&2\\0&1&4&1\\1&0&8&6\end{bmatrix}\sim\ -R_1+R_3\to R_3\begin{bmatrix}1&-1&3&2\\0&1&4&1\\0&1&5&4\end{bmatrix}\sim\ \begin{matrix}R_2+R_1\to R_1\\ \\ -R_2+R_3\to R_3\end{matrix}\begin{bmatrix}1&0&7&3\\0&1&4&1\\0&0&1&3\end{bmatrix}$$

$$\sim\ \begin{matrix}-7R_3+R_1\to R_1\\-4R_3+R_2\to R_2\\ \\ \end{matrix}\begin{bmatrix}1&0&0&-18\\0&1&0&-11\\0&0&1&3\end{bmatrix} = R$$

$$B=\begin{bmatrix}1&-1&4&5\\-2&1&-11&-8\\-1&2&2&2\end{bmatrix}\sim\ \begin{matrix} \\ 2R_1+R_2\to R_2\\R_1+R_3\to R_3\end{matrix}\begin{bmatrix}1&-1&4&5\\0&-1&-3&2\\0&1&6&7\end{bmatrix}$$

$$\sim\ \begin{matrix}-R_2+R_1\to R_1\\ \\ R_2+R_3\to R_3\end{matrix}\begin{bmatrix}1&0&7&3\\0&-1&-3&2\\0&0&3&9\end{bmatrix}$$

$$\sim\ \begin{matrix} \\ -R_2\to R_2\\ \frac{1}{3}R_3\to R_3\end{matrix}\begin{bmatrix}1&0&7&3\\0&1&3&-2\\0&0&1&3\end{bmatrix}$$

$$\sim\ \begin{matrix}-7R_3+R_1\to R_1\\-3R_3+R_2\to R_2\\ \\ \end{matrix}\begin{bmatrix}1&0&0&-18\\0&1&0&-11\\0&0&1&3\end{bmatrix}=R$$

b) From a.) $E_5E_4E_3E_2E_1A=R$ and $F_8F_7F_6F_5F_4F_3F_2F_1B=R$

where E_i and F_i are elementary matrices.

So $E_5E_4E_3E_2E_1A=F_8F_7F_6F_5F_4F_3F_2F_1B$

$$(E_5E_4E_3E_2E_1)^{-1}E_5E_4E_3E_2E_1A=(E_5E_4E_3E_2E_1)^{-1}F_8\cdots F_2F_1B$$

$$A=E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}F_8F_7F_6F_5F_4F_3F_2F_1B$$

$\therefore$ A is row equivalent to B.

Question 7. *(3 marks)* The augmented matrix of a linear system is given by

$$\begin{bmatrix} 1 & 3 & 1 & -4 & b_1 \\ 3 & -2 & 4 & 5 & b_2 \\ 4 & 1 & 5 & 1 & b_3 \\ 7 & -1 & 9 & 6 & b_4 \end{bmatrix}$$

Determine the restrictions on the b_i's for the system to be consistent.

$$\sim \begin{array}{l} \\ -3R_1+R_2\to R_2 \\ -4R_1+R_3\to R_3 \\ -7R_1+R_4\to R_4 \end{array} \begin{bmatrix} 1 & 3 & 1 & -4 & b_1 \\ 0 & -11 & 1 & 17 & b_2-3b_1 \\ 0 & -11 & 1 & 17 & b_3-4b_1 \\ 0 & -22 & 2 & 34 & b_4-7b_1 \end{bmatrix}$$

$$\sim \begin{array}{l} \\ \\ -R_2+R_3\to R_3 \\ -2R_2+R_4\to R_4 \end{array} \begin{bmatrix} 1 & 3 & 1 & -4 & b_1 \\ 0 & -11 & 1 & 17 & b_2-3b_1 \\ 0 & 0 & 0 & 0 & b_3-b_2-b_1 \\ 0 & 0 & 0 & 0 & b_4-2b_2-b_1 \end{bmatrix}$$

So consistent if $b_3-b_2-b_1=0$

and $b_4-2b_2-b_1=0$

Bonus Question.

Given

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix}$$

where the entries of the matrix are elements of $\mathbb{Z}_3$. Operations on the elements of $\mathbb{Z}_3$ can be defined by the following Cayley tables:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

a. *(2 marks)* Find the inverse of A, if possible.

b. *(3 marks)* Find all the solutions of

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

where $x_i \in \mathbb{Z}_3$ and the entries of the constant matrix are elements of $\mathbb{Z}_3$.

a) $$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \sim \begin{array}{l} \\ 2R_1+R_2\to R_2 \\ R_1+R_3\to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 2 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 1 \end{array}\right]$$

$$\sim \begin{array}{l} \\ \\ 2R_2+R_3\to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 2 & 1 \end{array}\right]$$

$\therefore$ A is not invertible

b) $$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 0 & 2 & 0 \end{array}\right] \sim \begin{array}{l} \\ 2R_1+R_2\to R_2 \\ R_1+R_3\to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 2 & 2 & 1 \end{array}\right]$$

$$\sim \begin{array}{l} 2R_2+R_1\to R_1 \\ \\ 2R_2+R_3\to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim 2R_2\to R_2 \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\to x_3=t,$

$t\in\mathbb{Z}_3$

$x_1=-t$

$x_1=2t$

$x_2=2-t$

$x_2=2+2t$

$t=0$

$(x_1,x_2,x_3)=(0,2,0)$

$t=1$

$(x_1,x_2,x_3)=(2,1,1)$

$t=2$

$(x_1,x_2,x_3)=(1,0,2)$