Dawson College: Linear Algebra: 201-NYC-05-S07: Fall 2015

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Test 2

This test is graded out of 46 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Given

$$A=\begin{bmatrix}0&1&1&1\\ \frac{1}{2}&\frac{1}{2}&1&\frac{1}{2}\\ \frac{2}{3}&\frac{1}{3}&\frac{1}{3}&0\\ -\frac{1}{3}&\frac{2}{3}&0&0\end{bmatrix},\ B=\begin{bmatrix}1&1&1&1&9&3&4&1&0&9\\0&2&1&1&4&9&2&7&7&9\\0&0&3&1&1&4&9&2&7&7\\0&0&0&4&1&1&4&9&2&7\\0&0&0&0&5&1&1&4&9&2\\0&0&0&0&0&1&1&1&4&9\\0&0&0&0&0&0&2&1&1&4\\0&0&0&0&0&0&0&3&1&1\\0&0&0&0&0&0&0&0&4&1\\0&0&0&0&0&0&0&0&0&5\end{bmatrix}$$

a. *(4 marks)* Evaluate the determinant of the matrix A by reducing the matrix to row echelon form (or triangular form).

b. *(4 marks)* If C is a 10×10 invertible matrix then evaluate $\det(C^{101}\operatorname{adj}(B)(C^{-1})^{101})$.

c. *(2 marks)* Solve for the component x_{10} of the solution of the system $Bx=b$ where

$$b=\begin{bmatrix}1\\2\\3\\4\\5\\6\\7\\8\\9\\10\end{bmatrix}$$

b)

$$\det(B)=1\cdot2\cdot3\cdot4\cdot5\cdot1\cdot2\cdot3\cdot4\cdot5=(5!)^2$$

$$\det(C^{101}\operatorname{adj}(B)(C^{-1})^{101})$$
$$=\det(C^{101})\det(\operatorname{adj}B)\det(C^{-1})^{101}$$
$$=\det(C^{101})(\det B)^{10-1}\det(C^{101})^{-1}$$
$$=\cancel{\det(C^{101})}\frac{1}{\cancel{\det C^{101}}}((5!)^2)^9=(5!)^{18}$$

a)

$$A\sim\begin{matrix}\\2R_2\to R_2\\3R_3\to R_3\\3R_4\to R_4\end{matrix}\begin{bmatrix}0&1&1&1\\1&1&2&1\\2&1&1&0\\-1&2&0&0\end{bmatrix}\sim R_1\leftrightarrow R_2\begin{bmatrix}1&1&2&1\\0&1&1&1\\2&1&1&0\\-1&2&0&0\end{bmatrix}$$

$$\sim\begin{matrix}\\ \\-2R_1+R_3\to R_3\\R_1+R_4\to R_4\end{matrix}\begin{bmatrix}1&1&2&1\\0&1&1&1\\0&-1&-3&-2\\0&3&2&1\end{bmatrix}$$

$$\sim\begin{matrix}\\ \\R_2+R_3\to R_3\\-3R_2+R_4\to R_4\end{matrix}\begin{bmatrix}1&1&2&1\\0&1&1&1\\0&0&-2&-1\\0&0&-1&-2\end{bmatrix}$$

$$\sim\begin{matrix}\\ \\ \\-\frac{1}{2}R_3+R_4\to R_4\end{matrix}\begin{bmatrix}1&1&2&1\\0&1&1&1\\0&0&-2&-1\\0&0&0&-\frac{3}{2}\end{bmatrix}=D$$

$$(2(3)(3)(-1))\det A=\det B$$
$$\det A=-1/6$$

c)

$$x_{10}=\frac{|B_{10}|}{|B|}=\frac{1\cdot2\cdot3\cdot4\cdot5\cdot1\cdot2\cdot3\cdot4\cdot10}{(5!)^2}$$
$$=\frac{\cancel{5!}\,\cancel{4!}\,10}{\cancel{5!}\,\cancel{4!}\,5}$$
$$=2$$

Question 2. Given

$$A = \begin{bmatrix} 4 & 3 & -2 & 8 \\ 2 & 3 & 1 & -4 \\ -1 & 2 & -1 & 4 \\ 1 & 2 & 2 & -8 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 & 6 \\ 2 & 3 & 0 & 1 \\ -3 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 2 & 6 \\ 2 & 3 & 0 & 1 \\ -2 & 2 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} = D$$

$C_3 + C_1 \to C_1 \quad C_3 + C_2 \to C_2$

a. *(5 marks)* Evaluate $\det(A)$ and $\det(B)$.

b. *(2 marks)* Prove or disprove: There exists C a square 4×4 invertible matrix such that $ABC = I$.

a) $\det(A) = 0$ since $C_4 = -4C_3$

$$\det(B) = \det(D) = (-1)^{4+3}(-1)\begin{vmatrix} 3 & 2 & 6 \\ 2 & 3 & 1 \\ -2 & 2 & 0 \end{vmatrix} \begin{matrix} 3 & 2 \\ 2 & 3 \\ -2 & 2 \end{matrix} = 2(1)(-2) + 6(2)(2) - 6(3)(-2) - (3)(1)(2)$$

$$= -4 + 24 + 36 - 6 = 50$$

b) there does not exist a matrix C since

$$\det(ABC) = \det I$$
$$\det A \det B \det C = 1$$
$$0 \cdot 50 \cdot \det C = 1$$
$$0 = 1 \quad ⚡$$

Question 3. [1] A square matrix A is said to be *idempotent* if $A^2 = A$.

a. *(2 marks)* Show that if A is idempotent then $\det(A) = 0$ or $\det(A) = 1$.

b. *(2 marks)* Show that if A is idempotent and $\det(A) = 1$ then $A = I$.

a) $\det A^2 = \det A$

$$0 = \det A^2 - \det A$$
$$0 = (\det A)^2 - \det A$$
$$0 = \det A(\det A - 1)$$

$\det A = 0 \qquad \det A - 1 = 0$

$\qquad\qquad\qquad \det A = 1$

b) $\det A = 1 \neq 0 \Rightarrow A$ is invertible

$$A^2 = A$$
$$AA = A$$
$$A^{-1}AA = A^{-1}A$$
$$IA = I$$
$$A = I$$

[1] From a John Abbott final examination

Question 4. *(5 marks)* Let

$$A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix}$$

and

$$B = \begin{bmatrix} 3d & 3e & 3f \\ a+2d & b+2e & c+2f \\ 4g & 4h & 4k \end{bmatrix}$$

If $\det(B) = 5$ then determine $\det(A)$.

So $\det A = \det A^T$

$$A^T = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \sim \begin{array}{c} 2R_2 + R_1 \to R_1 \\ 3R_2 \to R_2 \\ 4R_4 \to R_4 \end{array} \begin{bmatrix} a+2d & b+2e & c+2f \\ 3d & 3e & 3f \\ 4g & 4h & 4k \end{bmatrix}$$

$$\sim \quad R_1 \leftrightarrow R_2 \begin{bmatrix} 3d & 3e & 3f \\ a+2d & b+2e & c+2f \\ 4g & 4h & 4k \end{bmatrix}$$

$$3(4)(-1) \det A^T = \det B$$

$$\det A = \frac{-5}{12}$$

Question 5.[2] *(3 marks)* Prove: If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$ then

$$||\vec{u} + \vec{v}||^2 - ||\vec{u} - \vec{v}||^2 = 4\vec{u} \cdot \vec{v}$$

$LHS = (\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v}) - (\vec{u}-\vec{v}) \cdot (\vec{u}-\vec{v})$

$= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} - [\vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}]$

$+ \vec{v}\cdot\vec{v}$

$= 2\vec{u}\cdot\vec{v} + 2\vec{v}\cdot\vec{u}$

$= 2\vec{u}\cdot\vec{v} + 2\vec{u}\cdot\vec{v}$

$= 4\vec{u}\cdot\vec{v}$

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[2] From a John Abbott final examination

Question 6. Given

$$\begin{aligned}
\mathscr{A}&: \quad (-3,-2,-1) \\
\mathscr{L}&: \quad (x,y,z) = (1+t, 3+2t, 5-t) \quad t \in \mathbb{R} \\
\mathscr{P}&: \quad x+2y+3z = 4
\end{aligned}$$

a. *(2 marks)* Determine the equation of a plane parallel to $\mathscr{P}$ that passes through $\mathscr{A}$.

b. *([illegible] marks)* Determine the equation of a line perpendicular to $\mathscr{P}$ that passes through $\mathscr{A}$.

c. *(3 marks)* Determine the point on $\mathscr{L}$ that is closest to $\mathscr{A}$.

d. *(2 marks)* Determine the point on $\mathscr{P}$ that is closest to $\mathscr{A}$.

a) normal of plane $(1,2,3)$: $x+2y+3z=d$

sub A

$-3+2(-2)+3(-1)=d$

$-10=d$ $\quad \therefore x+2y+3z=-10$

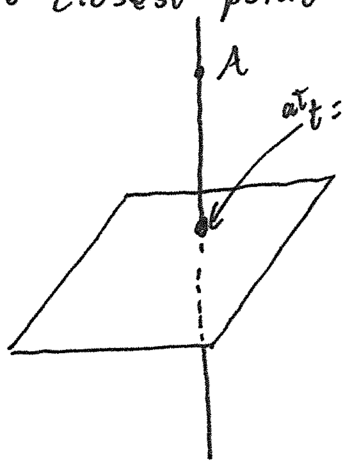
b) direction vector of line $(1,2,3)$: $(x,y,z)=(-3,-2,-1)+t(1,2,3)$ $t \in \mathbb{R}$

d) sub line from b in $\mathscr{P}$ $\quad (-3+t)+2(-2+2t)+3(-1+3t)=4$

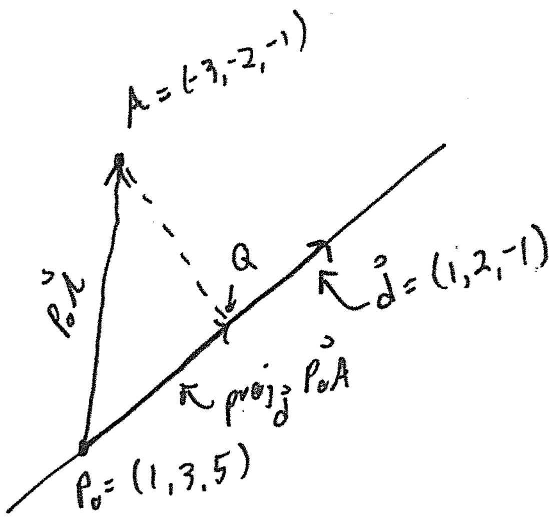
$-10+14t=4$

$t=+1$

$\therefore$ closest point is $(x,y,z)=(-3,-2,-1)+1(1,2,3)=(-2,0,2)$



c)



$\vec{P_0A}=A-P_0=(-3,-2,-1)-(1,3,5)$
$=(-4,-5,-6)$

$$\begin{aligned}
Q &= P_0 + \text{proj}_{\vec{d}}\, \vec{P_0A} \\
&= (1,3,5) + \frac{\vec{P_0A}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}\,\vec{d} \\
&= (1,3,5) + \frac{(-4,-5,-6)\cdot(1,2,-1)}{(1,2,-1)\cdot(1,2,-1)}(1,2,-1) \\
&= (1,3,5) + \frac{-8}{6}(1,2,-1) \\
&= (1,3,5) + \frac{-4}{3}(1,2,-1) \\
&= \left(\frac{-1}{3}, \frac{1}{3}, \frac{19}{3}\right)
\end{aligned}$$

Question 7. Given $\vec{u} = (1, -2, 3)$ and $\vec{w} = (-4, 1, -5)$.

a. *(2 marks)* Find a vector of length 3 that is oppositely directed to $\vec{w}$.

b. *(2 marks)* Compute the angle between $\vec{u}$ and $\vec{v}$, if possible.

c. *(2 marks)* Sketch $\vec{u}$.

a) $-3\frac{\vec{u}}{\|\vec{w}\|} = \frac{-3}{\sqrt{(-4)^2+1^2+(-5)^2}}(-4,1,-5) = \frac{-3}{\sqrt{42}}(-4,1,-5)$

b) $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$

$\vec{u}\cdot\vec{v} = (1,-2,3)\cdot(-4,1,-5) = -4 -2 -15 = -21$

$\|\vec{u}\| = \sqrt{1^2+(-2)^2+3^2} = \sqrt{14}$

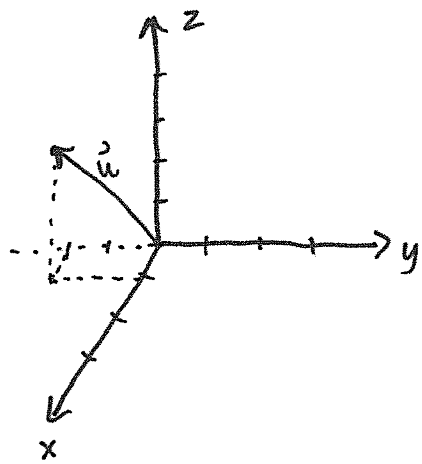
$\|\vec{w}\| = \sqrt{(-4)^2+1^2+(-5)^2} = \sqrt{42}$

$-21 = \sqrt{14}\sqrt{42}\cos\theta$

$\cos\theta = \frac{-21}{\sqrt{14}\sqrt{42}} = \frac{-21}{\sqrt{7}\sqrt{2}\sqrt{7}\sqrt{6}} = \frac{-3}{2\sqrt{3}} = -\frac{\sqrt{3}}{2}$

$\theta = 150^\circ$

c)



Bonus Question.*(5 marks)* note: $-1 = -1 \iff -1+0=-1$, similarly $-2 = 1$

$-1+3=-1$

$2=-1$

Given

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix} \sim \begin{matrix} 2R_1+R_2 \to R_2 \\ 2R_1+R_3 \to R_3 \end{matrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix} \sim R_2+R_3 \to R_3 \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} = B$$

$\det A = \det B = 1\cdot 2\cdot 1 = 2$

where the entries of the matrix are elements of $\mathbb{Z}_3$. Operations on the elements of $\mathbb{Z}_3$ can be defined by the following Cayley tables:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$C_{11} = \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2$ $\quad C_{13} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 = 2$

$C_{12} = (-1)\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$ $\quad C_{21} = (-1)\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 2(1) = 2$

Find the inverse of A using the adjoint of A, if possible.

$C_{22} = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 2$, $C_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$

$$A^{-1} = \frac{1}{|A|}\operatorname{adj} A = \frac{1}{2}\begin{bmatrix} 2 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}^T = 2\begin{bmatrix} 2 & 2 & 1 \\ 0 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$C_{13} = \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = 1$, $C_{32} = -\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = -2 = 1$

$C_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1 = 2$