Dawson College: **Linear Algebra**: 201-NYC-05-S07: Fall 2015

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Test 3

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Given

$$\begin{aligned} \mathscr{L}_1\!: \quad (x,y,z) &= (t, 1-t, -1+t) \quad t\in\mathbb{R} \\ \mathscr{L}_2\!: \quad (x,y,z) &= (1-2t, t, -1+2t) \quad t\in\mathbb{R} \end{aligned}$$

Let s be the parameter of $\mathcal{L}_2$

$(x,y,z) = (1-2s, s, -1+2s)$

a. *(7 marks)* Find the point on each line which is closest to the other.

b. *(2 marks)* Find the distance between the two lines.

a) $\mathcal{L}_1$ and $\mathcal{L}_2$ are not parallel since $\vec{d}_1 \neq k\vec{d}_2$ where $\vec{d}_1 = (1,-1,1)$

$\vec{d}_2 = (-2,1,2)$

Do the lines $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect? ① $t = 1-2s$

② $1-t = s$

③ $-1+t = -1+2s$

①+③ $-1+2t = 0$

$t = \frac{1}{2}$

sub into ①

$\frac{1}{2} = 1-2s$

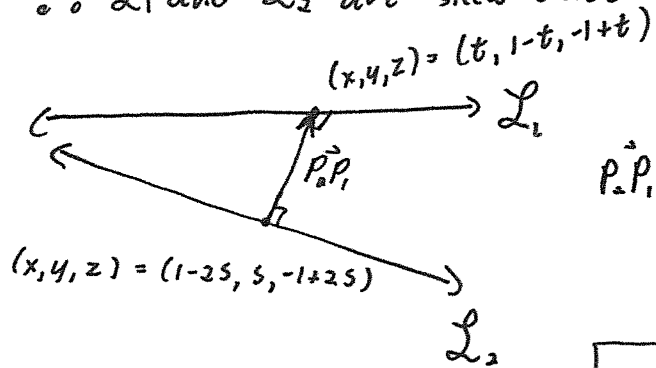
$s = \frac{1}{4}$

sub $t=\frac{1}{2}$, $s=\frac{1}{4}$ into ②

$1-\frac{1}{2} \stackrel{?}{=} \frac{1}{4}$ not consistent

$\therefore$ no intersection between $\mathcal{L}_1$ and $\mathcal{L}_2$.

$\therefore$ $\mathcal{L}_1$ and $\mathcal{L}_2$ are skew lines



$\vec{P_2P_1} = P_2 - P_1$

$= (1-2s, s, -1+2s) - (t, 1-t, -1+t)$

$= (1-2s-t, s+t-1, 2s-t)$

b) distance $= \left\| \left(1-2\left(\frac{7}{26}\right) - \frac{15}{26}, \frac{7}{26} + \frac{15}{26} - 1, 2\left(\frac{7}{26}\right) - \frac{15}{26}\right) \right\|$

$= \left\| \left(\frac{-3}{26}, \frac{-4}{26}, \frac{-1}{26} \right) \right\|$

$= \frac{1}{26}\sqrt{26}$

$$\begin{cases} \vec{d}_1 \cdot \vec{P_2P_1} = 0 \\ \vec{d}_2 \cdot \vec{P_2P_1} = 0 \end{cases}$$

$$\begin{cases} (1,-1,1)\cdot(1-2s-t, s+t-1, 2s-t) = 1-2s-t-s-t+1+2s-t = 0 \\ (-2,1,2)\cdot(1-2s-t, s+t-1, 2s-t) = -2+4s+2t+s+t-1+4s-2t = 0 \end{cases}$$

$$\begin{cases} ① \; 2-s-3t = 0 \\ ② \; -3+9s+t = 0 \end{cases}$$

3② + ①

$-7 + 26s = 0$

$s = \frac{7}{26}$

sub into ①

$t = 15/26$

Point on $\mathcal{L}_2$ $(x,y,z) = \left(1-2\left(\frac{7}{26}\right), \frac{7}{26}, -1+2\left(\frac{7}{26}\right)\right) = \left(\frac{6}{13}, \frac{7}{26}, \frac{-6}{13}\right)$

Point on $\mathcal{L}_1$ $(x,y,z) = \left(\frac{15}{26}, 1-\frac{15}{26}, -1+\frac{15}{26}\right) = \left(\frac{15}{26}, \frac{11}{26}, \frac{-11}{26}\right)$

Question 2. *(3 marks)* Prove or disprove: If $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions of the nonhomogeneuous linear system $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x}_1 - \mathbf{x}_2$ is a solution of the corresponding homogeneuous linear system $A\mathbf{x} = 0$.

Prove: Let x_1, x_2 be solutions of $Ax=b$, i.e. $Ax_1 = b$
$Ax_2 = b$

$A(x_1 - x_2)$
$= Ax_1 - Ax_2$
$= b - b$
$= 0$

$\therefore$ $x_1 - x_2$ is a solution of the corresponding linear system.

Question 3. *(4 marks)* Determine the area of the triangle generated by the vectors $\vec{u}$ and $\vec{v}$ if $\vec{u} \cdot (\vec{v} \times (\vec{u} \times \vec{v})) = 1936$

$$1936 = \vec{u} \cdot (\vec{v} \times (\vec{u} \times \vec{v})) = -(\vec{u} \times \vec{v}) \cdot (\vec{v} \times \vec{u})$$
$$= +(\vec{u} \times \vec{v}) \cdot (\vec{u} \times \vec{v})$$
$$= \|\vec{u} \times \vec{v}\|^2$$

So $\|\vec{u} \times \vec{v}\|^2 = 1936$
$\|\vec{u} \times \vec{v}\| = 44$

$\therefore$ area of triangle $\dfrac{\|\vec{u} \times \vec{v}\|}{2} = \dfrac{44}{2} = 22$

Question 4. Let $\vec{u} = (1, \lambda, -\lambda)$, $\vec{v} = (-2\lambda, -2, 2\lambda)$ and $\vec{w} = (\lambda - 2, -5\lambda - 2, -2)$.

a. *(2 marks)* For what value(s) of λ will $\{\vec{u}, \vec{v}\}$ be linearly dependent.

b. *(3 marks)* For what value(s) of λ will $\{\vec{u}, \vec{v}, \vec{w}\}$ be linearly independent.

a) $\vec{u}$ and $\vec{v}$ are linearly dependent iff they are multiples of each other $\vec{u} = K\vec{v}$

$(1, \lambda, -\lambda) = K(-2\lambda, -2, 2\lambda)$ $\therefore$ linearly dependent iff $\lambda = 1$

① $1 = -2K\lambda$
② $\lambda = -2K$
③ $-\lambda = 2K\lambda$

① + ③
$1 - \lambda = 0$
$1 = \lambda$

sub $\lambda = 1$ into ①
$1 = -2K(1)$
$-\frac{1}{2} = K$

sub λ, K into ②
$1 \stackrel{?}{=} -2(-\frac{1}{2})$
$1 = 1$ ✓

b) $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent iff their scalar triple product is not 0.

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 1 & \lambda & -\lambda \\ -2\lambda & -2 & 2\lambda \\ \lambda-2 & -5\lambda-2 & -2 \end{vmatrix} \underset{-(\lambda-2)R_1 + R_3 \to R_3}{\overset{2\lambda R_1 + R_2 \to R_2}{=}} \begin{vmatrix} 1 & \lambda & -\lambda \\ 0 & 2\lambda^2 - 2 & 2\lambda - 2\lambda^2 \\ 0 & -5\lambda - 2 - (\lambda-2)\lambda & -2 + \lambda(\lambda - 2) \end{vmatrix}$$

$$= (2\lambda^2 - 2)(-2 + \lambda(\lambda-2)) - (2\lambda - 2\lambda^2)((\lambda - 2) - (\lambda-2)\lambda)$$
$$= 2(\lambda-1)(\lambda+1)(\lambda^2 - 2\lambda - 2) - 2\lambda(1-\lambda)(-\lambda^2 - 3\lambda - 2)$$
$$= 2(\lambda-1)(\lambda+1)(\lambda^2 - 2\lambda - 2) - 2\lambda(\lambda-1)(\lambda^2 + 3\lambda + 2)$$
$$= 2(\lambda-1)(\lambda+1)(\lambda^2 - 2\lambda - 2) - 2\lambda(\lambda-1)(\lambda+2)(\lambda+1)$$
$$= 2(\lambda-1)(\lambda+1)[\lambda^2 - 2\lambda - 2 - \lambda(\lambda+2)]$$
$$= 2(\lambda-1)(\lambda+1)(-4\lambda - 2)$$

$\therefore$ linearly independent iff $\lambda \neq 1, -1, -\frac{1}{2}$

Question 5.[1] Let $\vec{u}$, $\vec{v}$, $\vec{p}$, and $\vec{q}$ be non-zero vectors in $\mathbb{R}^3$ and suppose $\text{span}(\{\vec{u}, \vec{v}\}) = \text{span}(\{\vec{p}, \vec{q}\})$

a. *(2 marks)* If $\text{span}(\{\vec{u}, \vec{v}\})$ is a line explain why $(\vec{u} \times \vec{v}) \times (\vec{p} \times \vec{q}) = \vec{0}$.

b. *(2 marks)* If $\text{span}(\{\vec{u}, \vec{v}\})$ is a plane explain why $(\vec{u} \times \vec{v}) \times (\vec{p} \times \vec{q}) = \vec{0}$.

a) $\{\vec{u}, \vec{v}\}$ spans a line then $\vec{u} = K\vec{v}$, then $\vec{u} \times \vec{v} = K\vec{v} \times \vec{v} = K(\vec{v} \times \vec{v}) = \vec{0}$

then $\vec{0} \times (\vec{p} \times \vec{q}) = \vec{0}$

b) $\{\vec{u}, \vec{v}\}$ and $\{\vec{p}, \vec{q}\}$ span the same plane through the origin, so we can obtain their respective normal likewise:

$\vec{n}_1 = \vec{u} \times \vec{v}$ and $\vec{n}_2 = \vec{p} \times \vec{q}$

Since the two set span the same plane $\vec{n}_1 = K\vec{n}_2$

So $(\vec{u} \times \vec{v}) \times (\vec{p} \times \vec{q}) = \vec{n}_1 \times \vec{n}_2$
$= K\vec{n}_2 \times \vec{n}_2$
$= K(\vec{n}_2 \times \vec{n}_2)$
$= K\vec{0}$
$= \vec{0}$

[1] From a John Abbott final examination

Question 6. Let $V = \{A \mid A \in \mathscr{M}_{3\times3}$ and A has a RREF with 2 or less leading ones$\}$.

a. *(2 marks)* Does V satisfy closure under vector addition? Justify.

b. *(2 marks)* Does V contain a zero vector? If so find it. Justify.

c. *(2 marks)* Does V satisfy closure under scalar multiplication? Justify.

d. *(2 marks)* Is V a subspace of $\mathscr{M}_{3\times3}$? Justify.

a) V is not closed under vector addition.

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in V$

$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in V$ since its RREF is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$A + B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \notin V$ since its RREF has 3 leading ones.

b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in V$ since it has no leading ones

c) Let $A \in V$, since $A \in V$, $A \sim$ Gauss Jordan $\sim R$ where R is in RREF and has 2 or less leading ones.

Then $kA \in V$ for any $k \in \mathbb{R}$ since $kA \underset{\substack{\frac{1}{k}R_2 \to R_2 \\ \frac{1}{k}R_3 \to R_3}}{\overset{\frac{1}{k}R_1 \to R_1}{\sim}} A \sim$ Gauss Jordan $\sim R$

has 2 or less leading ones

if $k \neq 0$

and $0A = 0 \in V$ by part b)

d) V is not a subspace of $M_{3\times3}$ since it is not closed under addition.

Question 7. *(7 marks)* $W = \{p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \mid p(1) = 0 \text{ and } p(-1) = 0\}$ a subspace of P_3. Find a basis B for W. Find the coordinate vector of $p(x) = -1 - x + x^2 + x^3$ relative to the basis B.

Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \in P_3$

If $p(1) = a_0 + a_1 + a_2 + a_3 = 0$
$p(-1) = a_0 - a_1 + a_2 - a_3 = 0$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 \end{bmatrix} \sim_{-R_1+R_2 \to R_2} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & -2 & 0 \end{bmatrix}$$

$$\sim_{-\frac{1}{2}R_2 \to R_2} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \sim_{-R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

So $a_0 = -a_2$
$a_1 = -a_3$

So $p(x) = -a_2 - a_3x + a_2x^2 + a_3x^3 \in W$
$= a_2\underbrace{(-1+x^2)}_{\in W} + a_3\underbrace{(-x+x^3)}_{\in W}$

$\therefore \beta = \{-1+x^2, -x+x^3\}$ spans W
and β is linearly independent since the two vectors are not a multiple of the other

$(-1-x+x^2+x^3)_\beta = (1,1)$ since $-1-x+x^2+x^3 = 1(-1+x^2)+1(-x+x^3)$

Bonus Question. *(5 marks)*
Given

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

where the entries of the matrix are elements of $\mathbb{Z}_3$. Operations on the elements of $\mathbb{Z}_3$ can be defined by the following Cayley tables:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

List all the vectors of the vector space $V = \{x \mid x \in \mathbb{Z}_3{}^3 \text{ and } Ax = 0\}$ over $\mathbb{Z}_3$.

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \sim \begin{matrix} R_1+R_2 \to R_2 \\ 2R_3 \to R_3 \end{matrix} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$$\sim \; R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \; R_2+R_1 \to R_1 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_3 = t \qquad t \in \mathbb{Z}_3$$

$$x_1 = 0$$

$$x_2 = -2t = t$$

$$\therefore \quad (x_1, x_2, x_3) = (0, t, t) = t(0,1,1) \qquad t \in \mathbb{Z}_3$$

$$W = \operatorname{span}(\{(0,1,1)\})$$

$$= \{(0,0,0), (0,1,1), (0,2,2)\}$$