Dawson College: **Linear Algebra**: 201-105-04-S03: Fall 2016

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Test 1

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1.

a. *(6 marks)* Solve the following system by Gauss-Jordan elimination:

$$\begin{array}{rcrcrcrcrcr} & & y & - & z & + & w & - & v & = & 0 \\ 2x & - & 3y & + & 4z & - & 4w & + & v & = & 0 \\ 3x & - & 3y & + & 4z & - & 4w & + & v & = & 0 \\ 5x & - & 5y & + & 7z & - & 7w & + & v & = & 0 \end{array}$$

b. *(1 mark)* Find two particular non-trivial solutions to the above system.

$$\begin{bmatrix} 0 & 1 & -1 & 1 & -1 & 0 \\ 2 & -3 & 4 & -4 & 1 & 0 \\ 3 & -3 & 4 & -4 & 1 & 0 \\ 5 & -5 & 7 & -7 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{matrix} R_1 \leftrightarrow R_2 \\ \\ \\ \\ \end{matrix} \begin{bmatrix} 2 & -3 & 4 & -4 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 3 & -3 & 4 & -4 & 1 & 0 \\ 5 & -5 & 7 & -7 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{matrix} \\ \\ 2R_3 \to R_3 \\ 2R_4 \to R_4 \end{matrix} \begin{bmatrix} 2 & -3 & 4 & -4 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 6 & -6 & 8 & -8 & 2 & 0 \\ 10 & -10 & 14 & -14 & 2 & 0 \end{bmatrix}$$

$$\sim \begin{matrix} \\ \\ -3R_1+R_3 \to R_3 \\ -5R_1+R_4 \to R_4 \end{matrix} \begin{bmatrix} 2 & -3 & 4 & -4 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 3 & -4 & 4 & -1 & 0 \\ 0 & 5 & -6 & 6 & -3 & 0 \end{bmatrix}$$

$$\sim \begin{matrix} \\ \\ -3R_2+R_3 \to R_3 \\ -5R_2+R_4 \to R_4 \end{matrix} \begin{bmatrix} 2 & -3 & 4 & -4 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 2 & 0 \\ 0 & 0 & -1 & 1 & 2 & 0 \end{bmatrix}$$

$$\sim \begin{matrix} 4R_3+R_1 \to R_1 \\ -R_3+R_2 \to R_2 \\ \\ -R_3+R_4 \to R_4 \end{matrix} \begin{bmatrix} 2 & -3 & 0 & 0 & 9 & 0 \\ 0 & 1 & 0 & 0 & -3 & 0 \\ 0 & 0 & -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{matrix} 3R_2+R_1 \to R_1 \\ \\ \\ \\ \end{matrix} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 & 0 \\ 0 & 0 & -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{matrix} \frac{1}{2}R_1 \to R_1 \\ \\ -R_3 \to R_3 \\ \\ \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $w = s$, $v = t$

$x = 0$
$y = 3t$
$z = 2t + s$ $\quad s, t \in \mathbb{R}$
$w = s$
$v = t$

b) $s=0$ $t=1$
$(x, y, z, w, v) = (0, 3, 2, 0, 1)$

$s=1$ $t=0$
$(x, y, z, w, v) = (0, 0, 1, 1, 0)$

Question 2. Consider the matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}, D = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}, E = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

a. *(2 marks)* Evaluate if possible, justify.

$$\text{trace}(C)AB = 1 \cdot \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -4 & -1 & 4 \\ 0 & -11 & 8 \\ 5 & -7 & 1 \end{bmatrix}$$

b. *(2 marks)* Evaluate if possible, justify.

AA $\quad A_{3\times 2} \; A_{3\times 2}$ is not defined since the # of columns of A is not equal to the # of rows of A.

c. *(2 marks)* Evaluate if possible, justify.

$$3DE - 2E = 3\begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -21 \end{bmatrix}$$

e. *(5 marks)* Solve for X if possible.

$$C((DX)^T - 2I)^{-1} = C$$

$$C^{-1}C((DX)^T - 2I)^{-1} = C^{-1}C$$

$$I((DX)^T - 2I)^{-1} = I$$

$$[((DX)^T - 2I)^{-1}]^{-1} = I^{-1}$$

$$(DX)^T - 2I = I$$

$$(DX)^T = I + 2I$$

$$(DX)^T = 3I$$

$$((DX)^T)^T = (3I)^T$$

$$DX = 3I$$

$$D^{-1}DX = D^{-1}(3I)$$

$$IX = 3D^{-1}I$$

$$X = 3D^{-1}$$

$$X = 3\frac{1}{-1}\begin{bmatrix} -3 & -1 \\ -4 & -1 \end{bmatrix}$$

$$X = \begin{bmatrix} 9 & 3 \\ 12 & 3 \end{bmatrix}$$

Question 3. *(2 marks)* A square matrix A is said to be idempotent if $A^2 = A$. Show that if A is idempotent, then $2A - I$ is invertible and its own inverse.

Premise: $A^2 = A$

conclusion: $(2A - I)(2A - I) = I$

$$\begin{aligned} LHS &= (2A-I)(2A-I) \\ &= 4A^2 - 2A - 2A + I \\ &= 4A^2 - 4A + I \\ &= 4A - 4A + I \quad \text{by premise} \\ &= I = RHS \end{aligned}$$

Question 4. *(3 marks)* Given an $n \times n$ identity I and $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m$ an arbitrary polynomial of degree m. Find the condition for which $p(I)$ is invertible.

$$\begin{aligned} p(I) &= a_0 I + a_1 I + a_2 I^2 + \cdots + a_m I^m \\ &= a_0 I + a_1 I + a_2 I + \cdots + a_m I \\ &= (a_0 + a_1 + \cdots + a_m) I \end{aligned}$$

For $P(I)$ to be invertible the RREF of $P(I)$ needs to be equal to I. $\therefore a_0 + a_1 + \cdots + a_m \neq 0$

Question 5. Consider the following augmented matrix of a system of linear equations.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & a^2-1 & ba-b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & (a-1)(a+1) & b(a-1) \end{bmatrix}$$

For which value(s) of a and b, if any, the system

a. *(2 marks)* has a unique solution, justify.

b. *(2 marks)* has no solutions, justify.

c. *(2 marks)* has infinitely many solutions, justify.

a) # leading 1 = # var:

$(a-1)(a+1) \neq 0$

$a - 1 \neq 0 \quad a + 1 \neq 0$

$a \neq 1 \quad a \neq -1$

b) Leading 1 in constant column:

$(a^2-1) = 0$ and $ba - b \neq 0$

$b(a-1) \neq 0$

$(a-1)(a+1) = 0$

$a = 1 \quad a = -1$

$b \neq 0 \quad a \neq 1$

$\therefore a = -1$

c) # leading 1 < # var:

$a^2 - 1 = 0$ and $ba - b = 0$

$(a-1)(a+1) = 0$

$a = 1 \quad a = -1$

$b(a-1) = 0$

$b = 0 \quad a = 1$

$\therefore a = 1$

Question 6. Consider the following system:

$$\begin{array}{rcrcrcl} x & + & y & & & = & 1 \\ x & + & y & + & z & = & 2 \\ & & y & + & z & = & 3 \end{array}$$

a. *(1 mark)* Write the above system as a matrix equation.

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{b}$$

b. *(3 marks)* Find the inverse of the coefficient matrix.

$[A \mid I]$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim -R_1 + R_2 \to R_2 \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim R_2 \leftrightarrow R_3 \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right]$$

$$\sim -R_3 + R_2 \to R_2 \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right]$$

$$\sim -R_2 + R_1 \to R_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right]$$

since we obtain the identity

$$A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

c. *(1 mark)* Solve the matrix equation by using the inverse of the coefficient matrix.

$$x = A^{-1}b = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

d. *(4 marks)* Express the coefficient matrix as a product of elementary matrices.

From b) $E_4 E_3 E_2 E_1 A = I \Rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$

E_1^{-1}:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim R_1 + R_2 \to R_2 \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1^{-1}$$

E_2^{-1}:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_2^{-1}$$

E_3^{-1}:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim R_3 + R_2 \to R_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = E_3^{-1}$$

E_4^{-1}:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim R_2 + R_1 \to R_1 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_4^{-1}$$

Question 7. *(3 marks)* Given that A, B and C are invertible matrices simplify the following expression.

$B^TA^T((AB)^T)^{-1}C(C^TA)^TB(CB)^{-1}$

$= B^TA^T(B^TA^T)^{-1}CA^T(C^T)^T\underbrace{BB^{-1}}_{I}C^{-1}$

$= B^T\underbrace{A^T(A^T)^{-1}}_{I}(B^T)^{-1}CA^T\underbrace{CC^{-1}}_{I}$

$= \underbrace{B^T(B^T)^{-1}}_{I}CA^T$

$= CA^T$

Bonus Question. *(3 marks)*
Prove: If c is a scalar and A is an $n \times n$ matrix then $\mathrm{tr}(cA^T) = c\mathrm{tr}(A)$.

Let $A = [a_{ij}]$

$LHS = tr[cA^T)$

$= tr(c[a_{ij}]^T)$

$= tr(c[a_{ji}])$

$= tr([ca_{ji}])$

$= ca_{11} + ca_{22} + \cdots + ca_{nn}$

$= c(a_{11} + a_{22} + \cdots + a_{nn}]$

$= c\,tr(A) = RHS$