Dawson College: **Linear Algebra**: 201-105-04-S03: Fall 2016

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Test 2

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Given

$$A=\begin{bmatrix}10&9&1&1&9&3&4&1&0&9\\0&9&1&1&4&9&2&7&7&9\\0&0&8&1&1&4&9&2&7&7\\0&0&0&7&1&1&4&9&2&7\\0&0&0&0&6&1&1&4&9&2\\0&0&0&0&0&5&1&1&4&9\\0&0&0&0&0&0&4&1&1&4\\0&0&0&0&0&0&0&3&1&1\\0&0&0&0&0&0&0&0&2&1\\0&0&0&0&12&2&2&8&18&4\end{bmatrix},\quad B=\begin{bmatrix}a&b\\c&d\end{bmatrix},\quad C=\begin{bmatrix}b&2a-3b\\d&2c-3d\end{bmatrix},\quad D=\begin{bmatrix}0&0&-2\\0&2&2\\4&14&-8\end{bmatrix}$$

a. *(2 marks)* Evaluate det(A).

$\det(A) = 0$ since $R_2 = 2R_1$.

b. *(2 marks)* True or false: $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, justify?

False, by the thm TFAE (or Equivalence) since $\det A = 0$

c. *(2 marks)* Evaluate det(D).

$$D \sim R_1 \leftrightarrow R_3 \begin{bmatrix}4&14&-8\\0&2&2\\0&0&-2\end{bmatrix} = E$$

$(-1)\det D = \det E$

$(-1)\det D = -16$

$\det D = 16$

c. *(2 marks)* Determine the conditions on the b_i's, if any, in order to guarantee that the system

$$D\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

is consistent.

There are no conditions on the b_i's since $\det D \neq 0$ and by the thm. TFAE (or Equivalence) the system is consistent $\forall b$.

d. *(4 marks)* If det(C) = 5 then determine det(B).

$$B \sim C_1 \leftrightarrow C_2 \begin{bmatrix}b&a\\d&c\end{bmatrix} \sim 2C_2 \begin{bmatrix}b&2a\\d&2c\end{bmatrix} \sim -3C_1 + C_2 \to C_2 \begin{bmatrix}b&2a-3b\\d&2c-3d\end{bmatrix} = C$$

$(2)(-1)\det B = \det C$

$\det B = \frac{5}{-2}$

Question 2. Given

$$A = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 3 & 0 & 0 & 4 \\ 5 & 0 & 0 & 6 \\ 0 & 7 & 8 & 0 \end{bmatrix}$$

and a 4×4 matrix B such that $\det(B) = 101$.

a. *(4 marks)* Evaluate $\det(A)$. $= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$

$= 0C_{11} + C_{12} + 2C_{13} + 0C_{14}$

$= C_{12} + 2C_{13}$

$= (-1)^{1+2}\begin{vmatrix} 3 & 0 & 4 \\ 5 & 0 & 6 \\ 0 & 8 & 0 \end{vmatrix} + 2(-1)^{1+3}\begin{vmatrix} 3 & 0 & 4 \\ 5 & 0 & 6 \\ 0 & 7 & 0 \end{vmatrix}$

$= -1\left[a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}\right] + 2\left[a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}\right]$

$= -\left[0C_{31} + 8C_{32} + 0C_{33}\right] + 2\left[0C_{31} + 7C_{32} + 0C_{33}\right]$

$= -8C_{32} + 14C_{32}$

$= +8\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} - 14\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}$

$= -6\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = -6\left[3(6) - 4(5)\right] = -6\left[-2\right] = 12$

b. *(4 marks)* Evaluate $\det(2A^2B^3)$. $= 2^4 \det(A^2B^3)$

$= 2^4 \det(A^2)\det(B^3)$

$= 16(\det(A))^2(\det(B))^3$

$= 16(12)^2(101)^3$

c. *(4 marks)* Evaluate $\det(\operatorname{adj}((3A^{-1})^T))$. $= \left(\det(3A^{-1})^T\right)^{4-1}$

$= \left(\det(3A^{-1})\right)^3$

$= \left(3^4 \det A^{-1}\right)^3$

$= (3^4)^3\left(\frac{1}{\det A}\right)^3$

$= \frac{3^{12}}{(\det A)^3} = \frac{3^{12}}{(12)^3} = \frac{3^{12}}{3^3 4^3} = \frac{3^9}{4^3}$

Question 3. *(5 marks)* Find the inverse of A by using the adjoint of A.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{matrix} 1 & 0 \\ 0 & 0 \\ 0 & 2 \end{matrix}$$

$$\det(A) = 1(0)(0) + 0(1)(0) + 1(0)(2) - 1(0)(0) - 2(1)(1) - (0)(0)(0)$$

$$= -2$$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} = -2$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix} = 0$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} = 2$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = -2$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} = 0$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$$

$$= \frac{1}{-2} [\text{matrix of Cof}]^T$$

$$= -\frac{1}{2} \begin{bmatrix} -2 & 0 & 0 \\ 2 & 0 & -2 \\ 0 & -1 & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$$

Question 4. A square matrix A is called *skew-symmetric* if $A^T = -A$.

a. *(3 marks)* Prove: If A is an invertible skew-symmetric matrix, then A^{-1} is skew-symmetric.

Premise: A is skew symmetric
$A^T = -A$

Conclusion: $(A^{-1})^T = -(A^{-1})$

$$LHS = (A^{-1})^T$$
$$= (A^T)^{-1}$$
$$= (-A)^{-1} = \frac{1}{-1} A^{-1} = -A^{-1} = RHS$$

c. *(2 marks)* Prove: If A is an $n \times n$ skew-symmetric matrix where n is odd, then $\det(A) = 0$.

Premise: A is skew symmetric
$A^T = -A$

Conclusion: $\det A = 0$

$$A^T = -A$$
$$\det(A^T) = \det(-A)$$
$$\det A = (-1)^n \det A$$
$$\det A = -\det A \quad \text{since } n \text{ is odd}$$

$\therefore \det A = 0$

Question 5. *(6 marks)* Determine for which value(s) of λ, if any, the following system can be solved using Cramer's Rule.

$$\begin{array}{rcrcrcrcl} x & - & 2y & + & z & - & 2w & = & 5 \\ 3x & - & 4y & & & + & 3w & = & 6 \\ & & 5y & & & - & \lambda w & = & 7 \\ 2x & - & \lambda y & - & 2z & & & = & 8 \end{array}$$

$$\underbrace{\begin{bmatrix} 1 & -2 & 1 & -2 \\ 3 & -4 & 0 & 3 \\ 0 & 5 & 0 & -\lambda \\ 2 & -\lambda & -2 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}}_{b}$$

Can use Cramer's Rule if $\det(A) \neq 0$

$$\det A = \begin{matrix} \\ -3R_1+R_2 \to R_2 \\ \\ -2R_1+R_4 \to R_4 \end{matrix} \begin{vmatrix} 1 & -2 & 1 & -2 \\ 0 & 2 & -3 & 9 \\ 0 & 5 & 0 & -\lambda \\ 0 & -\lambda+4 & -4 & 4 \end{vmatrix}$$

$= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + a_{41}C_{41}$

$= 1\,C_{11} + 0\,C_{21} + 0\,C_{31} + 0\,C_{41}$

$= C_{11}$

$$= (-1)^{1+1} \begin{vmatrix} 2 & -3 & 9 \\ 5 & 0 & -\lambda \\ -\lambda+4 & -4 & 4 \end{vmatrix}$$

$= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$

$= 5\,C_{21} + 0\,C_{22} + (-\lambda)C_{23}$

$$= 5(-1)^{2+1}\begin{vmatrix} -3 & 9 \\ -4 & 4 \end{vmatrix} + 0 + (-\lambda)(-1)^{2+3}\begin{vmatrix} 2 & -3 \\ -\lambda+4 & -4 \end{vmatrix}$$

$= -5[-12 + 36] + \lambda[-8 - 3\lambda + 12]$

$= -5[24] + \lambda[3\lambda + 4]$

$= -3\lambda^2 + 4\lambda - 120 \neq 0$

$$\lambda \neq \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{(4)^2 - 4(3)(-120)}}{2(3)} = \frac{-4 \pm \sqrt{-1424}}{6}$$

no λ such that $\det(A) = 0$

$\therefore$ Cramer's Rule works $\forall \lambda \in \mathbb{R}$

Bonus Question. *(3 marks)*
Show that

$$\det(A) = \frac{1}{2}\begin{vmatrix} \operatorname{tr}(A) & 1 \\ \operatorname{tr}(A^2) & \operatorname{tr}(A) \end{vmatrix}$$

for every 2×2 matrix A.

Bonus Question. Show that

$$\det(A) = \frac{1}{2}\begin{vmatrix} \operatorname{tr}(A) & 1 \\ \operatorname{tr}(A^2) & \operatorname{tr}(A) \end{vmatrix}$$

for every 2×2 matrix A.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^2 = AA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$

$$tr(A) = a+d$$

$$tr(A^2) = a^2+bc+bc+d^2$$
$$= a^2+2bc+d^2$$

$$|A| = ad-bc$$

$$RHS = \frac{1}{2}\begin{vmatrix} trA & 1 \\ trA^2 & trA \end{vmatrix} = \frac{1}{2}\left[(trA)^2 - trA^2\right]$$
$$= \frac{1}{2}\left[(a+d)^2 - (a^2+2bc+d^2)\right]$$
$$= \frac{1}{2}\left[a^2+2ad+d^2-a^2-2bc-d^2\right]$$
$$= \frac{1}{2}\left[2ad-2bc\right] = ad-bc = |A| \quad \blacksquare$$