Dawson College: Linear Algebra: 201-105-04-S03: Fall 2016

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Test 3

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Given $\vec{u} = (-1, \lambda, -3)$, $A(1,0,1)$, $B(0,1,2)$ and $C(3,-2,1)$.

a. *(2 marks)* For which value(s) of λ, if any, $\vec{u}$ is parallel to $\vec{AB}$.

$\vec{AB} = B - A = (0,1,2) - (1,0,1) = (-1,1,1)$

$\vec{u} \parallel \vec{AB}$ iff $\vec{u} = k\vec{AB}$

$(-1,\lambda,-3) = k(-1,1,1)$

$(-1,\lambda,-3) = (-k,k,k)$

$-1 = -k \qquad \lambda = k \qquad -3 = k$

$1 = k$

no solution!

$\therefore$ there does not exists values of λ such that $\vec{u}$ and $\vec{AB}$ are parallel.

b. *(2 marks)* For which value(s) of λ, if any, $\vec{u}$ is orthogonal to $\vec{AC}$.

$\vec{AC} = C - A = (3,-2,1) - (1,0,1) = (2,-2,0)$

$\vec{u} \perp \vec{AC}$ iff $\vec{u} \cdot \vec{AC} = 0$

$(-1,\lambda,-3)\cdot(2,-2,0) = 0$

$-2 - 2\lambda - 3(0) = 0$

$\lambda = -1$

$\therefore \vec{u} \perp \vec{AC}$ iff $\lambda = -1$

c. *(2 marks)* Compute $\text{proj}_{\vec{AB}\times\vec{AC}}(2,3,-4)$.

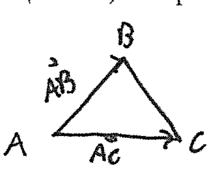
$$\vec{AB} \times \vec{AC} = \left(\begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix}, -\begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} \right) = (2,2,0)$$

$$\begin{matrix} -1 & 2 \\ 1 & -2 \\ 1 & 0 \end{matrix}$$

$$\text{proj}_{\vec{AB}\times\vec{AC}}(2,3,-4) = \frac{(\vec{AB}\times\vec{AC})\cdot(2,3,-4)}{(\vec{AB}\times\vec{AC})\cdot(\vec{AB}\times\vec{AC})}\,\vec{AB}\times\vec{AC}$$

$$= \frac{(2,2,0)\cdot(2,3,-4)}{(2,2,0)\cdot(2,2,0)}(2,2,0) = \frac{10}{8}(2,2,0) = \left(\frac{5}{2}, \frac{5}{2}, 0\right)$$

d. *(1 mark)* Compute the area of the triangle defined by A, B, C.



$$A = \frac{\|\vec{AB}\times\vec{AC}\|}{2} = \frac{\|(2,2,0)\|}{2} = \frac{\sqrt{4+4+0}}{2} = \frac{\sqrt{8}}{2} = \sqrt{2}$$

e. *(1 mark)* Compute the volume of the parallelepiped defined by $\vec{AC}$, $\vec{AB}$ and $\vec{v} = (1,0,0)$.

$$V = |\vec{v}\cdot(\vec{AB}\times\vec{AC})| = |(1,0,0)\cdot(2,2,0)| = |2| = 2$$

Question 2. *(5 marks)* Write the parametric equation of the line that passes through the point of intersection and orthogonal to both lines, where

$$\vec{x} = \begin{cases} x = t \\ y = -2+2t \\ z = 1+t \end{cases} \quad \text{and} \quad \vec{x} = \begin{cases} x = 2+s \\ y = 2-s \\ z = 3+2s \end{cases} . \quad s, t \in \mathbb{R}$$

$\mathcal{L}_1$ $\mathcal{L}_2$

$t = 2+s$ ①
$-2+2t = 2-s$ ②
$1+t = 3-2s$ ③

① + ②

$-2+3t = 4$
$3t = 6$
$t = 2$

sub into ①

$2 = 2+s$
$s = 0$

verify consistency sub into ③

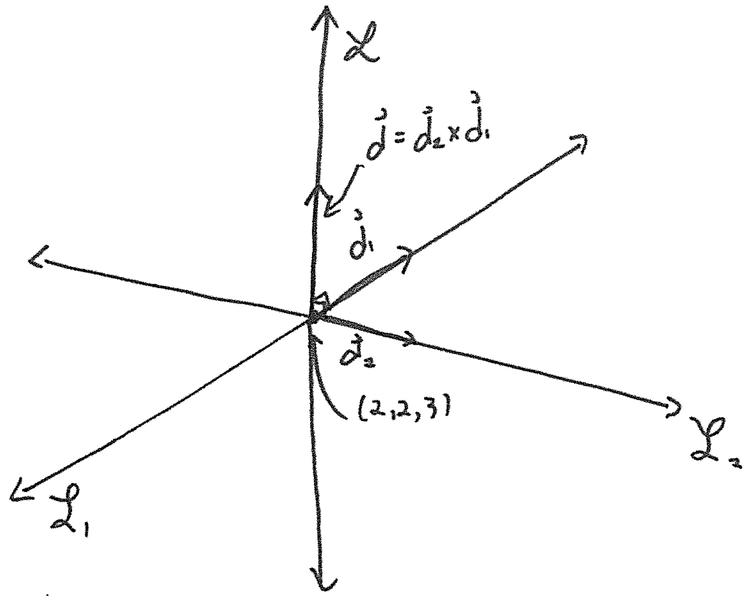
$1+2 \stackrel{?}{=} 3-2(0)$
$3 = 3$

$\therefore$ intersection when $s=0$, $t=2$

$\therefore$ the intersection is

$x = 2+0 = 2$
$y = 2-0 = 2$
$z = 3+2(0) = 3$

$(x,y,z) = (2,2,3)$



$$\vec{d} = \vec{d_2} \times \vec{d_1} = \left(\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} \right)$$

$$\begin{matrix} 1 & 1 \\ -1 & 2 \\ 2 & 1 \end{matrix} \quad = (-5, 1, 3)$$

$\mathcal{L}: (x,y,z) = (2,2,3) + t(-5,1,3)$

Question 3. *(5 marks)* Find the angle between $\vec{u} = (1,2,3)$ and $\vec{v} = (1,0,1)$.

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$

$1(1) + 2(0) + 3(1) = \sqrt{1^2+2^2+3^2} \sqrt{1^2+0^2+1^2} \cos\theta$

$\cos\theta = \dfrac{4}{\sqrt{2}\sqrt{14}}$

$\cos\theta = \dfrac{2}{\sqrt{7}}$

$\theta = \arccos\left(\dfrac{2}{\sqrt{7}}\right)$

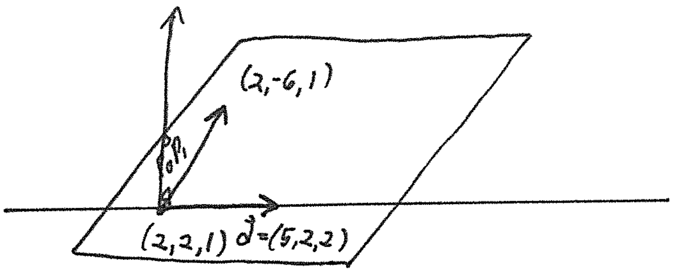
$\doteq 41°$

Question 4. *(5 marks)* Give the equation of the plane that contains the point $(2,-6,1)$ and the line $(x,y,z)=(2+5t,2+2t,1+2t)$. where $t \in \mathbb{R}$

$= (2,2,1) + t(5,2,2)$

$= P_0 + t\vec{d}$

Let $P_1(2,-6,1)$



$\vec{P_0P_1} = P_1 - P_0 = (2,-6,1)-(2,2,1)$

$= (0,-8,0)$

Parametric equation of plane

$(x,y,z) = (2,2,1) + s(0,-8,0) + t(5,2,2)$

where $s,t \in \mathbb{R}$

general equation of plane:

$\vec{n} = \vec{P_0P_1} \times \vec{d} = \left(\begin{vmatrix} -8 & 2 \\ 0 & 2 \end{vmatrix}, -\begin{vmatrix} 0 & 5 \\ 0 & 2 \end{vmatrix}, \begin{vmatrix} 0 & 5 \\ -8 & 2 \end{vmatrix} \right)$

$\begin{matrix} 0 & 5 \\ -8 & 2 \\ 0 & 2 \end{matrix} \quad = (-16, 0, 40)$

$ax+by+cz=d$

$-16x+0y+40z=d$

sub in $(2,-6,1)$

$-16(2)+0(-6)+40(1)=d$

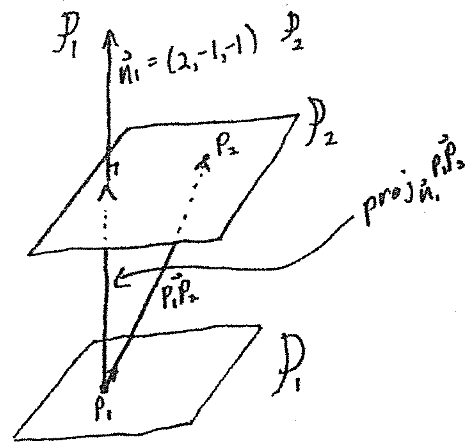
$8=d$

$-16x+40z=8$

$-2x+5z=1$

Question 5. *(5 marks)* Using projections find the distance between the given parallel planes.

$\underbrace{2x-y-z=5}_{P_1}$ and $\underbrace{-4x+2y+2z=12}_{P_2}$



$\vec{P_1P_2} = P_2 - P_1$

$= (0,6,0)-(0,0,-5)$

$= (0,6,5)$

$\text{proj}_{\vec{n}_1} \vec{P_1P_2} = \dfrac{\vec{P_1P_2}\cdot\vec{n}_1}{\vec{n}_1\cdot\vec{n}_1}\vec{n}_1$

$= \dfrac{(0,6,5)\cdot(2,-1,-1)}{(2,-1,-1)\cdot(2,-1,-1)}(2,-1,-1)$

$= \dfrac{-11}{4+1+1}(2,-1,-1)$

$= \dfrac{-11}{6}(2,-1,-1)$

Lets find a point on P_1: let $x=y=0$

$2(0)-0-z=5$

$z=-5 \quad P_1(0,0,-5)$

Lets find a point on P_2: let $x=z=0$

$-4(0)+2y+2(0)=12$

$y=6 \quad P_2(0,6,0)$

$d = \|\text{proj}_{\vec{n}_1}\vec{P_1P_2}\|$

$= \left\| \frac{-11}{6}(2,-1,-1) \right\|$

$= \frac{11}{6}\|(2,-1,-1)\| = \frac{11}{6}\sqrt{2^2+(-1)^2+(-1)^2}$

$= \frac{11\sqrt{6}}{6}$

Question 6. *(5 marks)* Maximize $Z = 2x + y + 3z$ subject to the constraints

$2x - y + z \le 100$
$x + y + 2z \le 70.$

make into equalities

$$\begin{aligned} 2x - y + z + S_1 \quad &= 100 \\ x + y + 2z \quad + S_2 &= 70 \end{aligned}$$

and we rewrite the objective function as: $-2x - y - 3z + Z = 0$

$$\begin{bmatrix} 2 & -1 & 1 & 1 & 0 & 0 & 100 \\ 1 & 1 & 2 & 0 & 1 & 0 & 70 \\ -2 & -1 & -3 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$r_1 = 100/1 = 100$
$r_2 = 70/2 = 35$ ← smallest ratio ∴ pivot row

↑ pivot column

$$\tfrac{1}{2}R_2 \to R_2 \begin{bmatrix} 2 & -1 & 1 & 1 & 0 & 0 & 100 \\ 1/2 & 1/2 & 1 & 0 & 1/2 & 0 & 35 \\ -2 & -1 & -3 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{array}{l} -R_2 + R_1 \to R_1 \\ \\ 3R_2 + R_3 \to R_3 \end{array} \begin{bmatrix} 3/2 & -3/2 & 0 & 1 & -1/2 & 0 & 65 \\ 1/2 & 1/2 & 1 & 0 & 1/2 & 0 & 35 \\ -1/2 & 1/2 & 0 & 0 & 3/2 & 1 & 105 \end{bmatrix}$$

$r_1 = 65/\tfrac{3}{2} = \frac{130}{3}$ ← smallest ratio ∴ pivot row
$r_2 = 35/\tfrac{1}{2} = 70$

↑ pivot column

$$\tfrac{2}{3}R_1 \to R_1 \begin{bmatrix} 1 & -1 & 0 & 2/3 & -1/3 & 0 & \frac{130}{3} \\ 1/2 & 1/2 & 1 & 0 & 1/2 & 0 & 35 \\ -1/2 & 1/2 & 0 & 0 & 3/2 & 1 & 105 \end{bmatrix}$$

$$\begin{array}{l} \\ -\tfrac{1}{2}R_1 + R_2 \to R_2 \\ \tfrac{1}{2}R_1 + R_3 \to R_3 \end{array} \begin{bmatrix} 1 & -1 & 0 & 2/3 & -1/3 & 0 & \frac{130}{3} \\ 0 & 1 & 1 & -1/3 & 2/3 & 0 & 40/3 \\ 0 & 0 & 0 & 1/3 & 4/3 & 1 & 380/3 \end{bmatrix}$$

$$\begin{aligned} \therefore x &= \frac{130}{3} \\ y &= 0 \\ z &= 40/3 \\ S_1 &= 0 \\ S_2 &= 0 \\ Z &= \frac{380}{3} \end{aligned}$$

Question 7. *(5 marks)* Minimize $Z = x + y$ subject to the constraints

$$x + y \geq 2$$
$$3x + y \geq 4.$$

We have

$$\begin{aligned} x + y - s_1 \quad\quad &= 2 \\ 3x + y \quad - s_2 &= 4 \\ x + y \quad\quad + W &= 0 \end{aligned}$$

$$\begin{array}{c} * \\ * \\ \\ \end{array} \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 2 \\ 3 & 1 & 0 & -1 & 0 & 4 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{array}{l} r = \frac{2}{1} = 2 \\ r = \frac{4}{3} \leftarrow \text{p.r.} \\ \\ \end{array}$$

$\uparrow$ p.c.

$$\frac{1}{3}R_2 \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 2 \\ 1 & 1/3 & 0 & -1/3 & 0 & 4/3 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{array}{r} -R_2 + R_1 \to R_1^{*} \\ \\ -R_2 + R_3 \to R_3 \end{array} \begin{bmatrix} 0 & 2/3 & -1 & 1/3 & 0 & 2/3 \\ 1 & 1/3 & 0 & -1/3 & 0 & 4/3 \\ 0 & 2/3 & 0 & 1/3 & 1 & -4/3 \end{bmatrix} \begin{array}{l} r = \frac{2/3}{2/3} = 1 \\ r = 4/3 / 1/3 = 4 \\ \\ \end{array}$$

$\uparrow$ p.c.

Let $W = -Z = -x - y$

$$\frac{3}{2}R_1 \begin{bmatrix} 0 & 1 & -3/2 & 1/2 & 0 & 1 \\ 1 & 1/3 & 0 & -1/3 & 0 & 4/3 \\ 0 & 2/3 & 0 & 1/3 & 1 & -4/3 \end{bmatrix}$$

$$\begin{array}{r} \\ -\frac{1}{3}R_1 + R_2 \to R_2 \\ -\frac{2}{3}R_1 + R_3 \to R_3 \end{array} \begin{bmatrix} 0 & 1 & -3/2 & 1/2 & 0 & 1 \\ 1 & 0 & 1/2 & -1/2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{bmatrix}$$

$\therefore$ $x = 1$ $\quad s_1 = 0$

$y = 1$ $\quad s_2 = 0$

$$Z = -W = -(-2) = 2$$

Question 8. If $\vec{u}$, $\vec{v}, \vec{w} \in \mathbb{R}^3$ and $\vec{u} \cdot (\vec{v} \times \vec{w}) = 2$ then evaluate

a. *(2 marks)*

$$\vec{u} \cdot (\vec{v} \times \vec{v}) = \vec{u} \cdot \vec{0} \quad \text{since } \vec{v} \times \vec{v} = 0$$

$$= 0$$

b. *(2 marks)*

$$(3\vec{v}) \cdot ((5\vec{u}) \times \vec{w}) = (3\vec{w}) \cdot (5\,(\vec{u} \times \vec{w})) = 5\,(3\vec{v}) \cdot (\vec{u} \times \vec{w}) = 15\,\vec{v} \cdot (\vec{u} \times \vec{w})$$

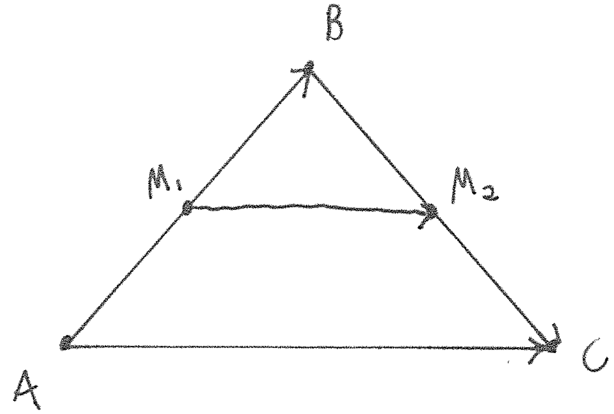
$$= 15 \begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= -15 \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \quad R_1 \leftrightarrow R_2$$

$$= -15(2)$$

$$= -30$$

Bonus Question. *(3 marks)*
Prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half as long.



Let M_1 be the midpoint of the line segment AB and M_2 be the midpoint of the line segment BC

We need to show that $\frac{1}{2}\vec{AC} = \vec{M_1M_2}$

Note that $\vec{M_1M_2} = \vec{M_1B} + \vec{BM_2}$

$= \frac{1}{2}\vec{AB} + \frac{1}{2}\vec{BC}$

$= \frac{1}{2}(\vec{AB} + \vec{BC})$

$= \frac{1}{2}\vec{AC}$