Dawson College: **Linear Algebra**: 201-105-04-S04: Fall 2016

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Test 1

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1.

a. *(6 marks)* Solve the following system by Gauss-Jordan elimination:

$$\begin{array}{rcrcrcrcrcr} & & y & - & z & + & w & - & v & = & 3 \\ 2x & - & 3y & + & 4z & - & 4w & + & v & = & 2 \\ 3x & - & 3y & + & 4z & - & 4w & + & v & = & 2 \end{array}$$

b. *(1 mark)* Find two particular solutions to the above system.

a) $\begin{bmatrix} 0 & 1 & -1 & 1 & -1 & 3 \\ 2 & -3 & 4 & -4 & 1 & 2 \\ 3 & -3 & 4 & -4 & 1 & 2 \end{bmatrix}$

$\begin{matrix} R_1 \leftrightarrow R_2 \\ \sim 2R_3 \to R_3 \end{matrix} \begin{bmatrix} 2 & -3 & 4 & -4 & 1 & 2 \\ 0 & 1 & -1 & 1 & -1 & 3 \\ 6 & -6 & 8 & -8 & 2 & 4 \end{bmatrix}$

$\sim -3R_1 + R_3 \to R_3 \begin{bmatrix} 2 & -3 & 4 & -4 & 1 & 2 \\ 0 & 1 & -1 & 1 & -1 & 3 \\ 0 & 3 & -4 & 4 & -1 & -2 \end{bmatrix}$

$\sim -3R_2 + R_3 \to R_3 \begin{bmatrix} 2 & -3 & 4 & -4 & 1 & 2 \\ 0 & 1 & -1 & 1 & -1 & 3 \\ 0 & 0 & -1 & 1 & 2 & -11 \end{bmatrix}$

$\begin{matrix} 4R_3 + R_1 \to R_1 \\ \sim -R_3 + R_2 \to R_2 \\ -R_3 \to R_3 \end{matrix} \begin{bmatrix} 2 & -3 & 0 & 0 & 9 & -42 \\ 0 & 1 & 0 & 0 & -3 & 14 \\ 0 & 0 & 1 & -1 & -2 & 11 \end{bmatrix}$

$\sim 3R_2 + R_1 \to R_1 \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 & 14 \\ 0 & 0 & 1 & -1 & -2 & 11 \end{bmatrix}$

$\sim \tfrac{1}{2}R_1 \to R_1 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 & 14 \\ 0 & 0 & 1 & -1 & -2 & 11 \end{bmatrix}$

Let $w = s$ $\quad s,t \in \mathbb{R}$
$v = t$

$x = 0$
$y = 14 + 3s$ $\quad s,t \in \mathbb{R}$
$z = 11 + 2t + s$

$\therefore (x,y,z,w,v) = (0, 14+3s, 11+2t+s, s, t$
$s,t \in \mathbb{R}$

b)

$s = t = 0 \quad (x,y,z,w,v) = (0,14,11,0,0)$
$s = 0 \ t = 1 \quad (x,y,z,w,v) = (0,14,13,0,1)$

Question 2. Consider the matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix},\ B = \begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix},\ D = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix},\ E = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

a. *(2 marks)* Evaluate if possible, justify.

$$\mathrm{trace}(C)A - \mathrm{trace}(D)B^T = (1)\begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix} - (-4)\begin{bmatrix} 2 & -3 \\ -5 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -10 \\ -17 & 10 \\ 9 & 3 \end{bmatrix}$$

b. *(2 marks)* Evaluate if possible, justify.

EA $\quad \underbrace{E_{2\times 1}\ A_{3\times 2}}$

\# columns of E not equal to # rows of A

$\therefore$ not defined

c. *(2 marks)* Evaluate if possible, justify.

$$(AC)E = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix}\left(\begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 17 \\ 31 \\ 2 \end{bmatrix}$$

e. *(4 marks)* Solve for X if possible.

$CXD = 10I_2$

$$C^{-1}CXDD^{-1} = C^{-1}10I_2D^{-1}$$

$$IXI = \frac{1}{-10}\begin{bmatrix} 0 & -2 \\ -5 & 1 \end{bmatrix}\cancel{10}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{-1}\begin{bmatrix} -3 & -1 \\ -4 & -1 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & -2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} -3 & -1 \\ -4 & -1 \end{bmatrix} = \begin{bmatrix} -8 & -2 \\ -19 & -6 \end{bmatrix}$$

n 3. *(5 marks)* Solve for X given that it satisfies

$$\left(2A+X^T\right)^{-1}=I$$

where $A=\begin{bmatrix}1 & 3\\ 1 & 2\end{bmatrix}$.

$$\left(\left(2A+X^T\right)^{-1}\right)^{-1} = I^{-1}$$

$$2A + X^T = I$$

$$X^T = I - 2A$$

$$X^T = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - 2\begin{bmatrix}1 & 3\\ 1 & 2\end{bmatrix}$$

$$X^T = \begin{bmatrix}-1 & -6\\ -2 & -3\end{bmatrix}$$

$$X = \begin{bmatrix}-1 & -2\\ -6 & -3\end{bmatrix}$$

Question 4. *(2 marks)* Let A and B be matrices such that AB is defined. Show that if the first and second columns of B are equal then so are the first and second columns of AB.

[1st column of AB] = A [1st column of B]

[2nd column of AB] = A [2nd column of B]

Since the first column and second column of B are the same then the 1st and 2nd column of AB are equal.

Question 5. *(3 marks)* Given an $n \times n$ matrix A such that $p(A) = 0$ where $p(x) = x^3 - x^2 + 1$. Determine the inverse of A in terms of A.

$0 = P(A)$

$0 = A^3 - A^2 + I$

$I = A^2 - A^3$

$I = A(A - A^2)$ $\quad$ $I = (A - A^2)A$

Question 6. Consider the following augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a nonzero number. Determine whether the system of the given augmented matrix is consistent. If consistent, is the solution unique?

a. *(2 marks)* $\left[\begin{array}{ccc|c} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{array}\right]$

all leading entries in variable columns
$\therefore$ consistent
$\because$ # leading entries = # var
$\therefore$ unique solution

b. *(2 marks)* $\left[\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & 0 & * \\ 0 & 0 & 0 & 0 & \blacksquare & 0 \\ 0 & 0 & 0 & 0 & * & \blacksquare \end{array}\right]$

If is equal to 0.
then the system is
inconsistent since
there is a leading in the constant column. If $(*)$ is nonzero
there is an elementary row operation to make the entry zero. Hence
the system is inconsistent.

Question 7. *(3 marks)* Given that A, H and M are invertible matrices simplify the following expression.

$$(AM^{-1})^{-1}(A^2)^T(H^TA^T(A^{-1})^T)^T = (M^{-1})^{-1}A^{-1}(AA)^T((A^{-1})^T)^T(A^T)^T(H^T)^T$$

$$= MA^{-1}A^TA^TA^{-1}AH$$

$$= MA^{-1}(A^T)^2 IH$$

$$= MA^{-1}(A^T)^2 H$$

Question 7. Consider the following system:

$$\begin{aligned} x - 2y &= 5 \\ 3x - 4y &= 6 \end{aligned}$$

a. *(1 mark)* Write the above system as a matrix equation.

$A\mathbf{x} = b$

$$\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

b. *(2 marks)* Solve the matrix equation by using the inverse of the coefficient matrix.

$$A\mathbf{x} = b$$
$$A^{-1}A\mathbf{x} = A^{-1}b$$
$$\mathbf{x} = A^{-1}b$$
$$= \frac{1}{2}\begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -8 \\ -9 \end{bmatrix} = \begin{bmatrix} -8/2 \\ -9/2 \end{bmatrix}$$

c. *(2 marks)* Give a geometrical interpretation of the solution set.

Two lines intersecting at $(x,y) = (-8/2, -9/2)$.

Question 8. *(1 mark)* True or False: a system with more unknowns than equations has at least one solution.

False, $\begin{aligned} x+y+z &= 1 \\ x+y+z &= 2 \end{aligned}$ has no solutions

Bonus Question. *(3 marks)* If c is a scalar and A is an $n \times n$ matrix then $tr(cA^T) = c\,tr(A)$

If $A = [a_{ij}]$ then

$$tr(c[a_{ij}]^T) = tr(c[a_{ji}]) = tr([ca_{ji}]) = ca_{11} + ca_{22} + \dots + ca_{nn}$$
$$= c(a_{11} + a_{22} + \dots + a_{nn})$$
$$= c\,tr(A)$$