Dawson College: **Linear Algebra**: 201-105-04-S04: Fall 2016

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Test 2

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Given

$$A = \begin{bmatrix} 10 & 9 & 1 & 1 & 9 & 3 & 4 & 1 & 0 & 9 \\ 0 & 9 & 1 & 1 & 4 & 9 & 2 & 7 & 7 & 9 \\ 0 & 0 & 8 & 1 & 1 & 4 & 9 & 2 & 7 & 7 \\ 0 & 0 & 0 & 7 & 1 & 1 & 4 & 9 & 2 & 7 \\ 0 & 0 & 0 & 0 & 6 & 1 & 1 & 4 & 9 & 2 \\ 0 & 0 & 0 & 0 & 0 & 5 & 1 & 1 & 4 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad C = \begin{bmatrix} c & d \\ -7a-5c & -7b-5d \end{bmatrix}, \quad D = \begin{bmatrix} 1 & -1 & -2 \\ -3 & 2 & 6 \\ 4 & 14 & -8 \end{bmatrix}$$

a. *(2 marks)* Evaluate $\det(A)$.

$\det(A) = (10)(9)(8)(7)(6)(5)(4)(3)(2)(1) = 10! = 3628800$ since the matrix is triangular

b. *(2 marks)* Is A invertible, justify?

A is invertible since $\det(A) \neq 0$

c. *(2 marks)* Evaluate $\det(D)$.

$\det(D) = 0$ since $C_3 = -2C_1$

c. *(2 marks)* Is D expressible as a product of elementary matrices, justify?

D is not expressible as a product of elementary matrices since $\det(D) = 0$ by TFAE (Equivalence Thm)

d. *(4 marks)* If $\det(C) = 2$ then determine $\det(B)$.

$$B \sim_{R_1 \leftrightarrow R_2} \begin{bmatrix} c & d \\ a & b \end{bmatrix} \sim_{-7R_2 \to R_2} \begin{bmatrix} c & d \\ -7a & -7b \end{bmatrix}$$

$$\sim_{-5R_1 + R_2 \to R_2} \begin{bmatrix} c & d \\ -7a-5c & -7b-5d \end{bmatrix} = C$$

(op. that change det) det [original matrix] = det [new matrix]

$$(-1)(-7) \det B = \det C$$

$$\det B = 2/7$$

Question 2. Given

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 0 \\ 0 & 5 & 6 & 0 \\ 7 & 0 & 0 & 8 \end{bmatrix}$$

a. *(4 marks)* Evaluate $\det(A)$.

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$$

$$= (1)(-1)^{1+1} \begin{vmatrix} 3 & 4 & 0 \\ 5 & 6 & 0 \\ 0 & 0 & 8 \end{vmatrix} + 0C_{12} + 0C_{13} + 2(-1)^{1+4} \begin{vmatrix} 0 & 3 & 4 \\ 0 & 5 & 6 \\ 7 & 0 & 0 \end{vmatrix}$$

$$= [a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}] - 2[a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

$$= [0C_{31} + 0C_{32} + 8(-1)^{3+3} \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}] - 2[7(-1)^{3+1} \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} + 0C_{32} + 0C_{33}]$$

$$= 8 \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} - 2[7 \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}]$$

$$= 8[3(6) - 4(5)] - 14[3(6) - 4(5)]$$

$$= -6[18 - 20]$$

$$= 12$$

b. *(4 marks)* Evaluate $\det(\operatorname{adj}((3A^T)^{-1}))$. $= (\det(3A^T)^{-1})^{4-1}$

$$= \left(\frac{1}{\det(3A^T)} \right)^3$$

$$= \left(\frac{1}{3^4 \det A^T} \right)^3$$

$$= \left(\frac{1}{3^4 \det A} \right)^3$$

$$= \left(\frac{1}{3^4 (12)} \right)^3$$

$$= \frac{1}{3^{12}\ 12^3}$$

Question 3. *(5 marks)* Express the following matrix as a product of elementary matrices

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}. \sim R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \tfrac{1}{2}R_2 \to R_2 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim -R_3 + R_1 \to R_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

So $E_3E_2E_1A = I$

$$(E_3E_2E_1)^{-1} E_3E_2E_1A = (E_3E_2E_1)^{-1} I$$

$$A = E_1^{-1}E_2^{-1}E_3^{-1}$$

E_1^{-1}:

$$I_3 \sim R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_1^{-1}$$

E_2^{-1}:

$$I_3 \sim 2R_2 \to R_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_2^{-1}$$

E_3^{-1}:

$$I_3 \sim R_3 + R_1 \to R_1 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3^{-1}$$

Question 4. Prove: If $A^TA = A$ then

a. *(2 marks)* A is symmetric, Need to show $A^T = A$

$$A^T = (A^TA)^T$$
$$= A^T(A^T)^T$$
$$= A^TA = A$$

$\therefore$ A is symmetric

b. *(2 marks)* $A = A^2$,

$$A = A^TA$$
$$= AA \quad \text{since } A \text{ is symmetric}$$
$$= A^2$$

$\therefore A = A^2$

c. *(2 marks)* $\det(A) = 0$ or $\det(A) = 1$.

$$A^TA = A$$
$$\det(A^TA) = \det(A)$$
$$\det(A^T)\det(A) = \det(A)$$
$$\det(A)\det(A) = \det(A)$$

$$(\det A)^2 = \det A$$
$$(\det A)^2 - (\det A) = 0$$
$$\det A(\det A - 1) = 0$$

$\det A = 0$ $\quad$ $\det A = 1$

Question 6. Consider the following system:

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 5 \\ & - & 4y & - & 2z & = & 6 \\ & & & & 3z & = & 1 \end{array}$$

a. *(1 mark)* Write the above system as a matrix equation.

$$\underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 0 & -4 & -2 \\ 0 & 0 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix}}_{b}$$

b. *(4 marks)* Find the inverse of the coefficient matrix using the adjoint.

$$\det A = 1(-4)(3) = -12$$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} -4 & -2 \\ 0 & 3 \end{vmatrix} = -12$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 0 & -2 \\ 0 & 3 \end{vmatrix} = 0$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 0 & -4 \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} -2 & 3 \\ 0 & 3 \end{vmatrix} = 6$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 0 & 3 \end{vmatrix} = 3$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & -2 \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} -2 & 3 \\ -4 & -2 \end{vmatrix} = 16$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 0 & -2 \end{vmatrix} = 2$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & -2 \\ 0 & -4 \end{vmatrix} = -4$$

$$\text{matrix of cofactors} = \begin{bmatrix} -12 & 0 & 0 \\ 6 & 3 & 0 \\ 16 & 2 & -4 \end{bmatrix}$$

$$\text{adj}(A) = [\text{matrix of cofactors}]^T = \begin{bmatrix} -12 & 6 & 16 \\ 0 & 3 & 2 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{\det A} \text{adj}(A) = \frac{1}{-12} \begin{bmatrix} -12 & 6 & 16 \\ 0 & 3 & 2 \\ 0 & 0 & -4 \end{bmatrix}$$

c. *(1 mark)* Solve the system using the inverse of the coefficient matrix.

$$\mathbf{x} = A^{-1} b$$

$$= \frac{1}{-12} \begin{bmatrix} -12 & 6 & 16 \\ 0 & 3 & 2 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix}$$

$$= \frac{1}{-12} \begin{bmatrix} -12(5) + 6(6) + 16 \\ 3(6) + 2(1) \\ -4(1) \end{bmatrix}$$

$$= \frac{1}{-12} \begin{bmatrix} -8 \\ 20 \\ -4 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -5/3 \\ 1/3 \end{bmatrix}$$

Question 8. *(3 marks)* Solve the following system using Cramer's Rule, if possible:

$$\begin{array}{rcrcr} x & - & 2y & = & 5 \\ 3x & - & 4y & = & 6 \end{array}$$

$$\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$A \quad \mathbf{x} = b$$

$$\det A = \begin{vmatrix} 1 & -2 \\ 3 & -4 \end{vmatrix} = -4 + 6 = 2$$

$$\det A_1 = \begin{vmatrix} 5 & -2 \\ 6 & -4 \end{vmatrix} = -20 + 12 = -8$$

$$\det A_2 = \begin{vmatrix} 1 & 5 \\ 3 & 6 \end{vmatrix} = 6 - 15 = -9$$

$$x = \frac{\det A_1}{\det A} = \frac{-8}{2} = -4$$

$$y = \frac{\det A_2}{\det A} = \frac{-9}{2}$$

Bonus Question. *(3 marks)*
Show that

$$\det(A) = \frac{1}{2}\begin{vmatrix} \operatorname{tr}(A) & 1 \\ \operatorname{tr}(A^2) & \operatorname{tr}(A) \end{vmatrix}$$

for every 2×2 matrix A.

Bonus Question. Show that

$$\det(A) = \frac{1}{2}\begin{vmatrix} \mathrm{tr}(A) & 1 \\ \mathrm{tr}(A^2) & \mathrm{tr}(A) \end{vmatrix}$$

for every 2×2 matrix A.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^2 = AA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$

$tr(A) = a + d$

$tr(A^2) = a^2 + bc + bc + d^2$

$= a^2 + 2bc + d^2$

$|A| = ad - bc$

$$RHS = \frac{1}{2}\begin{vmatrix} trA & 1 \\ trA^2 & trA \end{vmatrix} = \frac{1}{2}\left[(trA)^2 - trA^2\right]$$

$$= \frac{1}{2}\left[(a+d)^2 - (a^2 + 2bc + d^2)\right]$$

$$= \frac{1}{2}\left[a^2 + 2ad + d^2 - a^2 - 2bc - d^2\right]$$

$$= \frac{1}{2}\left[2ad - 2bc\right] = ad - bc = |A|$$ ■