Dawson College: **Linear Algebra**: 201-105-04-S04: Fall 2016

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Test 3

This test is graded out of 42 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Given $\vec{u} = (-3, \lambda, -1)$, $A(2, 0, 1)$, $B(1, 0, 2)$ and $C(3, -2, -1)$.

a. *(2 marks)* For which value(s) of λ, if any, $\vec{u}$ is parallel to $\vec{AB}$.

$\vec{AB} = B - A = (1,0,2) - (2,0,1) = (-1,0,1)$

$\vec{u} \parallel \vec{AB}$ iff $\vec{u} = k\vec{AB}$

$(-3,\lambda,-1) = k(-1,0,1)$

$(-3,\lambda,-1) = (-k,0,k)$

$-3 = -k \quad \lambda = 0 \quad -1 = k$

no solution! $\therefore$ there does not exists values of λ such that $\vec{u}$ and $\vec{AB}$ are parallel

b. *(2 marks)* For which value(s) of λ, if any, $\vec{u}$ is orthogonal to $\vec{AC}$.

$\vec{AC} = C - A = (3,-2,-1) - (2,0,1) = (1,-2,-2)$

$\vec{u} \perp \vec{AC}$ iff $\vec{u} \cdot \vec{AC} = 0$

$(-3,\lambda,-1)\cdot(1,-2,-2) = 0$

$-3 - 2\lambda + 2 = 0$

$-1 = 2\lambda$

$-\frac{1}{2} = \lambda$ $\quad \therefore \vec{u} \perp \vec{AC}$ iff $\lambda = -1/2$

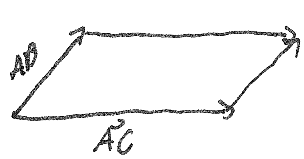
c. *(2 marks)* Compute $\text{proj}_{(2,3,-4)}\vec{AB} \times \vec{AC}$.

$\text{proj}_{(2,3,-4)} \vec{AB} \times \vec{AC} = \frac{(2,3,-4)\cdot\vec{AB}\times\vec{AC}}{(2,3,-4)\cdot(2,3,-4)}(2,3,-4) = \frac{(2,3,-4)\cdot(2,-1,2)}{(2,3,-4)\cdot(2,3,-4)}(2,3,-4) = \frac{-7}{29}(2,3,-4)$

$\vec{AB} \times \vec{AC} = \left(\begin{vmatrix} 0 & -2 \\ 1 & -2 \end{vmatrix}, -\begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix}, \begin{vmatrix} -1 & 1 \\ 0 & -2 \end{vmatrix} \right)$

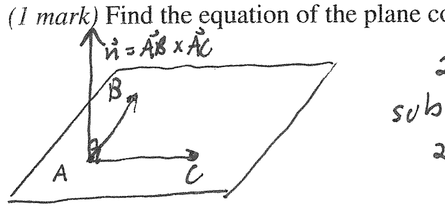
$\begin{matrix} -1 & 1 \\ 0 & -2 \\ 1 & -2 \end{matrix} \quad = (2,-1,2)$

d. *(1 mark)* Compute the area of the parallelogram defined by ~~[illegible]~~ $\vec{AB}$ and $\vec{AC}$



$A = \|\vec{AB} \times \vec{AC}\| = \|(2,-1,2)\| = \sqrt{9} = 3$

e. *(1 mark)* Find the equation of the plane containing the points A, B and C.



$2x - y + 2z = d$

sub B to solve for d.

$2(1) - 0 + 2(2) = d$

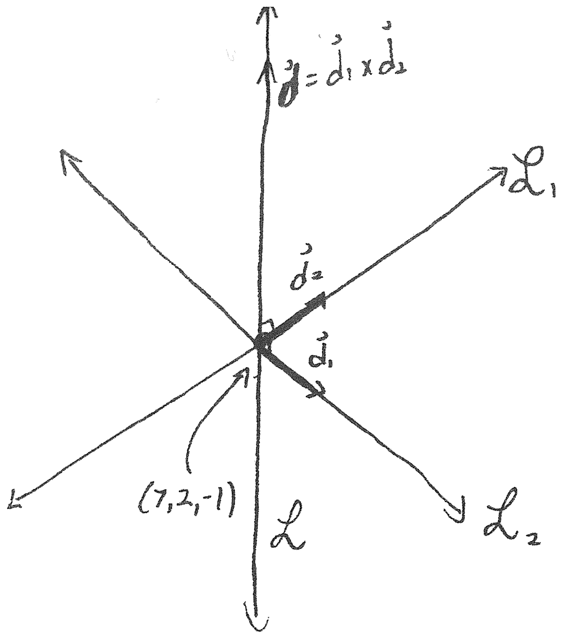
$6 = d$

$\therefore 2x - y + 2z = 6$

or

$(x,y,z) = (1,0,2) + s(-1,0,1) + t(1,-2,-2)$

Question 2. *(5 marks)* Write the parametric equation of the line that passes through the point of intersection and orthogonal of both lines, where $\vec{x} = (2,1,1) + t(5,1,-2)$, $t \in \mathbb{R}$ and $\vec{x} = (-2,-1,2) + s(3,1,-1)$, $s \in \mathbb{R}$.



① $2 + 5t = -2 + 3s$

② $1 + t = -1 + s$

③ $1 - 2t = 2 - s$

Solve for t by ②+③: $2 - t = 1$

$-t = -1$

$t = 1$ sub into ②

$1 + 1 = -1 + s$

$3 = s$

Check consistency by subbing into ①

$2 + 5(1) \stackrel{?}{=} -2 + 3(3)$

$7 = 7$

$\therefore$ the two lines intersect at $(x,y,z) = (2,1,1) + 1(5,1,-2)$

$= (7,2,-1)$

$\vec{d} = \vec{d_1} \times \vec{d_2} = \left(\begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix}, - \begin{vmatrix} 5 & 3 \\ -2 & -1 \end{vmatrix}, \begin{vmatrix} 5 & 3 \\ 1 & 1 \end{vmatrix} \right)$

$\begin{matrix} 5 & 3 \\ 1 & 1 \\ -2 & -1 \end{matrix} \quad = (1, -1, 2)$

$\therefore \quad \mathcal{L}: \quad (x, y, z) = (7, 2, -1) + t(1, -1, 2).$

Question 3. *(5 marks)* Find the angle between $\vec{u} = (3,2,1)$ and $\vec{v} = (0,2,0)$

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$

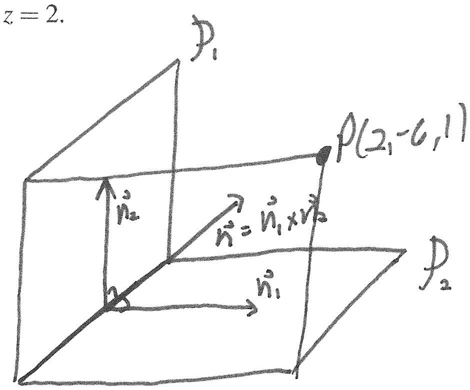
$3(0) + 2(2) + 1(0) = \sqrt{3^2 + 2^2 + 1^2} \sqrt{0^2 + 2^2 + 0^2} \cos\theta$

$4 = \sqrt{14} \sqrt{4} \cos\theta$

$\frac{2}{\sqrt{14}} = \cos\theta$

$\theta = \arccos\left(\frac{2}{\sqrt{14}}\right) = 57.69°$

Question 4. *(5 marks)* Give the equation of the plane that contains the point $(2,-6,1)$ and orthogonal to both planes: $x-y+z=1$ and $2x+y-z=2$.



$$\vec{n}=\vec{n}_1\times\vec{n}_2=\left(\begin{vmatrix}-1&1\\1&-1\end{vmatrix},-\begin{vmatrix}1&2\\1&-1\end{vmatrix},\begin{vmatrix}1&2\\-1&1\end{vmatrix}\right)$$

$$\begin{matrix}1&2\\-1&1\\1&-1\end{matrix}=(0,+3,3)$$

$$ax+by+cz=d$$
$$3y+3z=d$$

sub into equation to solve for d

$$3(-6)+3(1)=d$$
$$-15=d$$

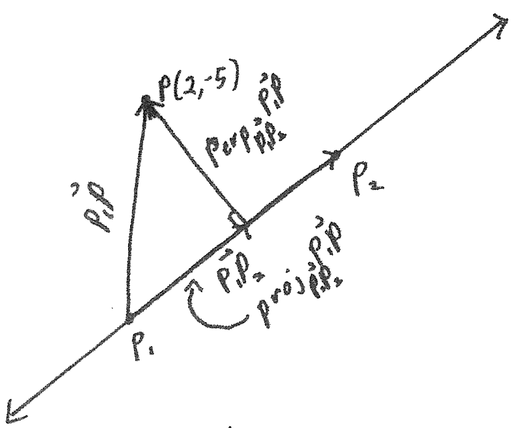
$$\therefore 3y+3z=-15$$
$$y+z=-5$$

or

$$(x,y,z)=(2,-6,1)+s(1,-1,1)+t(2,1,-1)$$

Question 5. *(5 marks)* Using projections find the distance between the point and the line.

$(2,-5)$; $y=-4x+2$



$$\vec{P_1P}=P-P_1=(2,-5)-(0,2)=(2,-7)$$
$$\vec{P_1P_2}=P_2-P_1=(1,-2)-(0,2)=(1,-4)$$

$$Perp_{\vec{P_1P_2}}\vec{P_1P}=\vec{P_1P}-proj_{\vec{P_1P_2}}\vec{P_1P}=\vec{P_1P}-\frac{\vec{P_1P}\cdot\vec{P_1P_2}}{\vec{P_1P_2}\cdot\vec{P_1P_2}}\vec{P_1P_2}$$
$$=(2,-7)-\frac{(2,-7)\cdot(1,-4)}{(1,-4)\cdot(1,-4)}(1,-4)$$
$$=(2,-7)-\frac{30}{17}(1,-4)$$
$$=\left(\frac{4}{17},\frac{1}{17}\right)$$

Let's find P_1: let $x=0$

$y=-4(0)+2=2$

$\therefore P_1(0,2)$

P_2: let $x=1$

$y=-4(1)+2=-2$

$\therefore P_2(1,-2)$

$$\text{distance}=\|Perp_{\vec{P_1P_2}}\vec{P_1P}\|$$
$$=\left\|\left(\frac{4}{17},\frac{1}{17}\right)\right\|$$
$$=\frac{1}{17}\|(4,1)\|=\frac{\sqrt{17}}{17}$$

Question 6. *(5 marks)* Maximize $Z = 3x + y$ subject to the constraints

$$2x - y \le 60$$
$$x + y \le 50.$$

convert to equalities

$$\begin{aligned} 2x - y + s_1 \quad &= 60 \\ x + y \quad + s_2 &= 50 \\ -3x - y \quad + Z &= 0 \end{aligned}$$

$$\begin{bmatrix} 2 & -1 & 1 & 0 & 0 & 60 \\ 1 & 1 & 0 & 1 & 0 & 50 \\ -3 & -1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$r_1 = 60/2 = 30 \leftarrow$ pivot row

$r_2 = 50/1 = 50$

pivot column

$$\sim \tfrac{1}{2}R_1 \to R_1 \begin{bmatrix} 1 & -1/2 & 1/2 & 0 & 0 & 30 \\ 1 & 1 & 0 & 1 & 0 & 50 \\ -3 & -1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{array}{l} \\ -R_1 + R_2 \to R_2 \\ 3R_1 + R_3 \to R_3 \end{array} \begin{bmatrix} 1 & -1/2 & 1/2 & 0 & 0 & 30 \\ 0 & 3/2 & -1/2 & 1 & 0 & 20 \\ 0 & -5/2 & 3/2 & 0 & 1 & 90 \end{bmatrix}$$

$\leftarrow$ pivot row

pivot column

$$\sim \tfrac{2}{3}R_2 \to R_2 \begin{bmatrix} 1 & -1/2 & 1/2 & 0 & 0 & 30 \\ 0 & 1 & -1/3 & 2/3 & 0 & 40/3 \\ 0 & -5/2 & 3/2 & 0 & 1 & 90 \end{bmatrix}$$

$$\sim \begin{array}{l} \tfrac{1}{2}R_2 + R_1 \to R_1 \\ \\ \tfrac{5}{2}R_2 + R_3 \to R_3 \end{array} \begin{bmatrix} 1 & 0 & 1/3 & 1/3 & 0 & \frac{110}{3} \\ 0 & 1 & -1/3 & 2/3 & 0 & 40/3 \\ 0 & 0 & 2/3 & 5/3 & 1 & \frac{370}{3} \end{bmatrix}$$

$x = \frac{110}{3}$

$y = 40/3$

$s_1 = 0$

$s_2 = 0$

$Z = \frac{370}{3}$

Question 7. *(5 marks)* Minimize $Z = 2x + y + z$ subject to the constraints

$$x + z \geq 2$$
$$2x + y + z \geq 3.$$

convert to equalities

$$\begin{aligned} x + z - s_1 &= 2 \\ 2x + y + z - s_2 &= 3 \\ 2x + y + z + C &= 0 \end{aligned}$$

$$\begin{aligned} C &= -Z \\ &= -(2x + y + z) \\ &= -2x - y - z \end{aligned}$$

pivot column

$$\begin{bmatrix} 1 & 0 & 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 1 & 0 & -1 & 0 & 3 \\ 2 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} r_1 = 2/1 = 2 \\ r_2 = 3/2 = 1.5 \leftarrow \text{pivot row} \end{array}$$

$$\tfrac{1}{2}R_2 \to R_2 \begin{bmatrix} 1 & 0 & 1 & -1 & 0 & 0 & 2 \\ 1 & 1/2 & 1/2 & 0 & -1/2 & 0 & 3/2 \\ 2 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

pivot column

$$\begin{array}{r} -R_2 + R_1 \to R_1 \\ \\ -2R_2 + R_3 \to R_3 \end{array} \begin{bmatrix} 0 & -1/2 & +1/2 & -1 & 1/2 & 0 & 1/2 \\ 1 & 1/2 & 1/2 & 0 & -1/2 & 0 & 3/2 \\ 0 & 0 & 0 & 0 & 1 & 1 & -3 \end{bmatrix} \quad \begin{array}{l} r_1 = \frac{1/2}{1/2} = 1 \leftarrow \text{pivot row} \\ r_2 = \frac{3/2}{1/2} = 3 \end{array}$$

$$2R_1 \to R_1 \begin{bmatrix} 0 & -1 & 1 & -2 & 1 & 0 & 1 \\ 1 & 1/2 & 1/2 & 0 & -1/2 & 0 & 3/2 \\ 0 & 0 & 0 & 0 & 1 & 1 & -3 \end{bmatrix}$$

$$-\tfrac{1}{2}R_1 + R_2 \to R_2 \begin{bmatrix} 0 & -1 & 1 & -2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -3 \end{bmatrix}$$

$x = 3/2$

$y = 0$

$z = 1$

$s_1 = 0$

$s_2 = 0$

$$\begin{aligned} C &= -Z \\ -C &= Z \\ -(-3) &= Z \\ 3 &= Z \end{aligned}$$

Question 8. If $\vec{u},\ \vec{v}, \vec{w} \in \mathbb{R}^3$ and $\vec{u} \cdot (\vec{v} \times \vec{w}) = 3$ then evaluate

a. *(2 marks)*

$$(\vec{u} - \vec{v}) \cdot (\vec{v} \times \vec{w}) = \vec{u} \cdot (\vec{v} \times \vec{w}) - \vec{v} \cdot (\vec{v} \times \vec{w}) \quad \text{by distributivity}$$

$$= 3 - \begin{vmatrix} v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

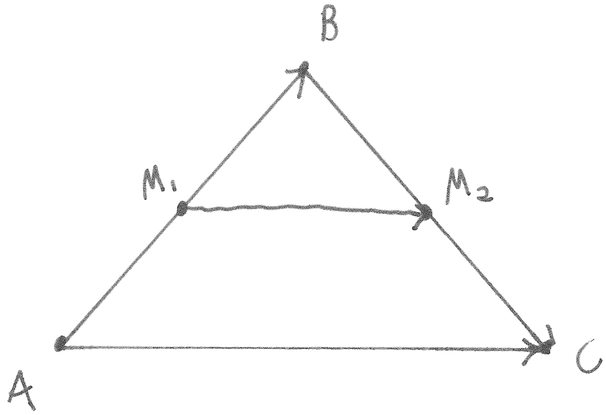
$$= 3 - 0 \qquad \text{since } R_1 = R_2$$

$$= 3$$

b. *(2 marks)*

$$(\vec{w} \times \vec{w}) \cdot \vec{v} = \vec{0} \cdot \vec{v} = 0$$

Bonus Question. *(3 marks)*
Prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half as long.



Let M_1 be the midpoint of the line segment AB and M_2 be the midpoint of the line segment BC

We need to show that $\frac{1}{2}\vec{AC} = \vec{M_1M_2}$

Note that $\vec{M_1M_2} = \vec{M_1B} + \vec{BM_2}$

$= \frac{1}{2}\vec{AB} + \frac{1}{2}\vec{BC}$

$= \frac{1}{2}(\vec{AB} + \vec{BC})$

$= \frac{1}{2}\vec{AC}$