Name: Y. Lamontagne

Test 1

This test is graded out of marks. No books, notes, watches or cell phones are allowed. You are only permitted to use the Sharp EL-531XG or Sharp EL-531X calculator. Give the work in full; – unless otherwise stated, reduce each answer to its simplest, exact form; – and write and arrange your exercise in a legible and orderly manner. If you need more space for your answer use the back of the page.

Question 1. Given

$$M = \begin{bmatrix} 1 & 2 & 2 & -1 & 3 & 1 \\ 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{matrix} -3R_3+R_1 \to R_1 \\ -2R_3+R_2 \to R_2 \end{matrix} \begin{bmatrix} 1 & 2 & 2 & -1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim -2R_2+R_1 \to R_1 \begin{bmatrix} 1 & 2 & 0 & -3 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where M is in row echelon form.

b) M, $M \sim R_3+R_2 \to R_2 \begin{bmatrix} 1 & 2 & 2 & -1 & 3 & -2 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

a. *(3 marks)* Find the reduced row echelon form of M.

b. *(2 marks)* Find 2 different row echelon form of M.

c. *(3 marks)* Find the solution set of the system of linear equations whose augmented matrix is M by using back substitution.

d. *(2 marks)* Find two particular solution of the system of linear equations whose augmented matrix is M by using back substitution.

e. *(2 marks)* Find the solution set of the homogeneous system of linear equations whose coefficient matrix is M.

f. *(2 marks)* Find a particular solution the homogeneous system of linear equations whose coefficient matrix is M when the solution of the first variable is equal to 1.

c. Let $x_2 = s$, $x_4 = t$, $s,t \in \mathbb{R}$ sub into $\begin{cases} x_1 + 2x_2 + 2x_3 - x_4 + 3x_5 = 1 \\ x_3 + x_4 + 2x_5 = 2 \\ x_5 = 1 \end{cases}$

$$\begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \begin{cases} x_1 + 2s + 2x_3 - t + 3x_5 = 1 \\ x_3 + t + 2x_5 = 2 \\ x_5 = 1 \end{cases} \quad \text{sub (3) into (1),(2)} \quad \begin{matrix} (4) \\ (5) \end{matrix} \begin{cases} x_1 + 2s + 2x_3 - t + 3(1) = 1 \\ x_3 + t + 2(1) = 2 \end{cases} \Rightarrow (6)\ x_3 = -t$$

sub (6) into (4) $x_1 + 2s + 2(-t) - t + 3 = 1 \Rightarrow x_1 = -2 + 3t - 2s$

$\therefore (x_1, x_2, x_3, x_4, x_5) = (-2 + 3t - 2s, s, -t, t, 1)$.

d) if $s = t = 0 \Rightarrow (x_1, x_2, x_3, x_4, x_5) = (-2, 0, 0, 0, 1)$, if $s = t = 1 \Rightarrow (x_1, x_2, x_3, x_4, x_5) = (-1, 1, -1, 1, 1)$

e) $[M|0] \sim \dots \sim \begin{bmatrix} 1 & 2 & 0 & -3 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Let $x_2 = s$, $x_4 = t$, $x_6 = u$, $s,t,u \in \mathbb{R}$

$$\begin{cases} x_1 + 2s - 3t + 2u = 0 \\ x_3 + t = 0 \\ x_5 + u = 0 \end{cases}$$

$$\begin{cases} x_1 = -2s + 3t - 2u \\ x_2 = s \\ x_3 = -t \\ x_4 = t \\ x_5 = -u \\ x_6 = u \end{cases} \quad s,t,u \in \mathbb{R}.$$

f) Let $s = u = 0$, $t = \frac{1}{3}$ then

$(x_1, x_2, x_3, x_4, x_5) = (1, 0, -\frac{1}{3}, \frac{1}{3}, 0)$.

Question 2. Consider the matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}, D = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}, E = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, F = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & -5 & -1 \end{bmatrix}$$

Evaluate the following if possible, justify.

a. *(2 marks)* trace$(D)F^2$ $= (-1+(-3)) \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & -5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & -5 & -1 \end{bmatrix} = -4 \begin{bmatrix} -1 & -13 & 4 \\ -1 & -12 & -5 \\ 5 & 5 & -9 \end{bmatrix}$

b. *(2 marks)* $(F - AB)^T$ $= \left(\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & -5 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix} \right)^T$

$$= \left(\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & -5 & -1 \end{bmatrix} - \begin{bmatrix} -4 & -1 & 4 \\ 0 & -11 & 8 \\ 5 & -7 & 1 \end{bmatrix} \right)^T = \left(\begin{bmatrix} 5 & 3 & -1 \\ -1 & 11 & -6 \\ -5 & 2 & -2 \end{bmatrix} \right)^T$$

$$= \begin{bmatrix} 5 & -1 & -5 \\ 3 & 11 & 2 \\ -1 & -6 & -2 \end{bmatrix}$$

c. *(2 marks)* trace$(F)FB$

$= (1+0+(-1))\, FB$

$= 0\, F_{3\times 3}\, B_{2\times 3}$

not defined since # of columns of F $\neq$ # of rows of B

Question 3. *(2 marks)* Write the 3×2 matrix $B = [b_{ij}]$ whose entries satisfy

$$b_{ij} = \begin{cases} 1 & \text{if } |i-j| \geq 1, \\ i+j & \text{if } |i-j| < 1. \end{cases}$$

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 4 \\ 1 & 1 \end{bmatrix}$$

Question 4. *(3 marks)* Show that every square matrix can be written as a sum of two invertible matrices. *(Hint: Write A as a sum of lower and upper triangular matrice.)*

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ a_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & l_{nn} \end{bmatrix} + \begin{bmatrix} U_{11} & a_{12} & \cdots & a_{1n} \\ 0 & U_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & U_{nn} \end{bmatrix} = L + U$

↑ lower triangular matrix ↑ upper triangular matrix

$l_{ii} = \frac{a_{ii}}{2}$ if $a_{ii} \neq 0$ or else $l_{ii} = 1$

$U_{ii} = \frac{a_{ii}}{2}$ if $a_{ii} \neq 0$ or else $U_{ii} = -1$

both L and U are invertible since all elements of the main diagonal are not equal to zero.

Question 4. *(3 marks)* Solve for A, if possible. A is a $\mathcal{M}_{2\times 2}$ diagonal matrix and satisfies $A^2 - 3A + 2I = 0$.

Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

$A^2 - 3A + 2I = 0$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^2 - 3\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} - \begin{bmatrix} 3a & 0 \\ 0 & 3b \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a^2 - 3a + 2 & 0 \\ 0 & b^2 - 3b + 2 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (a-2)(a-1) & 0 \\ 0 & (b-2)(b-1) \end{bmatrix}$

$\therefore A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$

Question 5. Determine whether the following statements are true or false for any $n \times n$ matrices A and B. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

a. *(3 marks)* If $A^3 = 0$ then $A^2 - A + I$ is invertible. *(Hint: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$.)*

By the factored sum of cubes identity we have

$$A^3 + I^3 = (A+I)(A^2 - AI + I^2)$$
$$0 + I = (A+I)(A^2 - A + I)$$
$$I = (A+I)(A^2 - A + I)$$

Hence, the statement is true and $A+I$ is the inverse of $A^2 - A + I$ if $A^3 = 0$

b. *(3 marks)* All elementary matrices are symmetric.

False,

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim 2R_2 + R_1 \to R_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = E$ is an elementary matrix

But $E^T \neq E$, therefore not symmetric.

Let A & B be nxn matrices.

Question 6. *(3 marks)* Prove or disprove: If $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have only the trivial solution then $(AB)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Premise: $Ax = 0$ and $Bx = 0$ only have the trivial solution

Conclusion: $(AB)x = 0$

Since $Ax=0$ and $Bx=0$ only have the trivial solution, A and B are invertible. It follows that AB is invertible since the product of two invertible matrix is invertible. Hence by the equivalence thm. $(AB)x = 0$ only has the trivial solution since AB is invertible.

Question 7. Given the following system $\begin{cases} 2x & = a \\ 3x + 4y & = b \\ 5x + 7y + 11z = c \end{cases}$

Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 5 & 7 & 11 \end{bmatrix}$.

a. *(5 marks)* Find the inverse of the coefficient matrix.

b. *(2 marks)* Express the inverse of the coefficient matrix as a product of elementary matrices, if possible. *(Only the final answer will be graded)*

c. *(2 marks)* For what value(s), if any, of a, b, c is the system consistent, justify.

d. *(2 marks)* Solve the system using the inverse for the value(s), if any, found in part b.

a) $$[A \mid I] \sim \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 & 1 & 0 \\ 5 & 7 & 11 & 0 & 0 & 1 \end{array}\right] \underset{\substack{2R_2 \to R_2 \\ 2R_3 \to R_3}}{\sim} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 6 & 8 & 0 & 0 & 2 & 0 \\ 10 & 14 & 22 & 0 & 0 & 2 \end{array}\right]$$

$$\underset{\substack{-3R_1 + R_2 \to R_2 \\ -5R_1 + R_3 \to R_3}}{\sim} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 8 & 0 & -3 & 2 & 0 \\ 0 & 14 & 22 & -5 & 0 & 2 \end{array}\right] \underset{8R_3 \to R_3}{\sim} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 8 & 0 & -3 & 2 & 0 \\ 0 & 8(14) & 8(22) & -40 & 0 & 16 \end{array}\right]$$

$$\underset{-14R_2 + R_3 \to R_3}{\sim} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 8 & 0 & -3 & 2 & 0 \\ 0 & 0 & 8(22) & 2 & -28 & 16 \end{array}\right] \underset{\substack{\frac{1}{2}R_1 \to R_1 \\ \frac{1}{8}R_2 \to R_2 \\ \frac{1}{8(22)}R_3 \to R_3}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -\frac{3}{8} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{88} & \frac{-7}{44} & \frac{1}{11} \end{array}\right]$$

b)

$$A^{-1} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -14 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{176} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c) The system is consistent $\forall a, b, c$ since A is invertible, by the equivalence theorem.

d) $A\mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$A^{-1}A\mathbf{x} = A^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$I\mathbf{x} = A^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\mathbf{x} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{3}{8} & \frac{1}{4} & 0 \\ \frac{1}{88} & \frac{7}{44} & \frac{1}{11} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{2}a \\ -\frac{3}{8}a + \frac{1}{4}b \\ \frac{1}{88}a - \frac{7}{44}b + \frac{1}{11}c \end{bmatrix}$$

(3 marks)

Bonus Question. Prove or disprove: If A is elementary then A^2 is elementary.

proof:

Case 1: Elementary matrix obtained by interchanging two rows.

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \sim R_i \leftrightarrow R_j \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} = E$$

$E^2 = EE = I$ mult. on the left by E same as interchanging R_i and R_j which result in I.

I is an elementary matrix, obtained by multiplying any row by 1.

Case 2: Elementary matrix obtained by multiplying a row by a non-zero constant.

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \sim cR_i \to R_i \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & c & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} = E$$

$$E^2 = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & c^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

which is an elementary matrix obtained by performing $c^2 R_i \to R_i$ on I.

Case 3: Elementary matrix obtained by adding a multiple of one row to another row.

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \sim cR_i + R_j \to R_j \begin{array}{c} \begin{array}{r} i^{th} \text{ column} \\ \downarrow \qquad \end{array} \\ \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & c & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \end{array} = E$$

$$E^2 = EE = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 2c & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

mult. on the left by E is equivalent to performing $cR_i + R_j \to R_j$ on E.

E^2 is an elementary matrix since it can be obtained by performing $2cR_i + R_j \to R_j$ on I.

$\therefore$ if E is an elementary matrix then so is E^2.