Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S2: Fall 2018

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Quiz 14

This quiz is graded out of 12 marks. No books, watches, notes or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1.[1] *(4 marks)* Given that $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are three linearly independent vectors in $\mathbb{R}^n$. For which value(s) of k will the vectors $\mathbf{u}+2\mathbf{v}$, $\mathbf{v}+3\mathbf{w}$ and $k\mathbf{u}+\mathbf{w}$ be linearly dependent?

$c_1(\underline{u}+2\underline{v}) + c_2(\underline{v}+3\underline{w}) + c_3(k\underline{u}+\underline{w}) = \underline{0}$

$(c_1+kc_3)\underline{u} + (2c_1+c_2)\underline{v} + (3c_2+c_3)\underline{w} = \underline{0}$

since $\underline{u}, \underline{v}$ and $\underline{w}$ are linearly independent, the only solution to the above is the trivial solution.

$$c_1 + kc_3 = 0 \qquad 2c_1 + c_2 = 0 \qquad 3c_2 + c_3 = 0 \qquad \begin{bmatrix} 1 & 0 & k \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $A\underline{c} = \underline{0}$ has only the trivial solution iff $|A| \neq 0$ by equivalence thm.

$$0 \neq 1\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} + k\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix}$$

$0 \neq 1 + 6k$

$\frac{-1}{6} \neq k \quad \therefore$ lin. ind. iff $k \neq -\frac{1}{6}$

Question 2. Determine whether the following statement is true or false. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

a. *(4 marks)* If $\overbrace{\mathbf{v}_1,\dots,\mathbf{v}_n}^{S}$ are linearly dependent nonzero vectors, then at least one vector $\mathbf{v}_k$ is a unique linear combination of $\mathbf{v}_1,\dots,\mathbf{v}_{k-1}$.

True, the subsets of S of the form $S_i = \{\underline{v}_1, \dots, \underline{v}_i\}$ are either linearly independent or linearly dependent. Let k be the smallest value for which S_k is linearly dependent. This implies $\exists c_i \neq 0$ and $c_k \neq 0$ s.t. $c_1\underline{v}_1 + \dots + c_{k-1}\underline{v}_{k-1} + c_k\underline{v}_k = \underline{0}$.* (note c_k must be $\neq 0$ or else it would imply that S_{k-1} is linearly dependent). From * we obtain $\underline{v}_k = -\frac{c_1}{c_k}\underline{v}_1 - \dots - \frac{c_{k-1}}{c_k}\underline{v}_k$. Now suppose that $\underline{v}_k$ does not have a unique linear combination of $v_1, \dots, v_{k-1}$. Then $\underline{v}_k = a_1\underline{c}_1 + \dots + a_j\underline{v}_j + \dots a_{k-1}\underline{v}_{k-1}$ and $\underline{v}_k = d_1\underline{v}_1 + \dots + d_j\underline{v}_j + \dots + d_{k-1}\underline{v}_{k-1}$ where $\exists l$ s.t. $a_l \neq b_l$ suppose one such l is j then making both linear combination equal and putting all terms on one side we obtain $(a_1-d_1)\underline{v}_1 + \dots + \underbrace{(a_j-d_j)}_{\neq 0}\underline{v}_j + \dots + (a_{k-1}-d_{k-1})\underline{v}_{k-1} = \underline{0}$. a non-trivial linear combination of $\underline{0}$. Hence S_{k-1} is linearly dependent. ↯ $\therefore$ the linear combination is unique.

b. *(3 marks)* The set of 2×2 matrices that contain exactly two 1's and two 0's is a linearly independent set in $\mathcal{M}_{2\times 2}$.

Let $M_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $M_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $M_5 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $M_6 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

The set $S = \{M_1, M_2, M_3, M_4, M_5, M_6\}$ is linearly dependent since the exist the following non-trivial linear combination

$$0M_1 + M_2 + 0M_3 + M_4 - M_5 - M_6 = 0$$

$\therefore$ False

[1] from a John Abbott Final Examination

Question 3.[2] *(1 mark each)*

I. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a linearly dependent set of vectors.
Select the best statement.

- A. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is always a linearly dependent set of vectors.
- B. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is always a linearly independent set of vectors.
- C. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ could be a linearly independent or linearly dependent set of vectors depending on the vectors chosen.
- D. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is a linearly independent set of vectors unless $\mathbf{u}_4$ is a linear combination of other vectors in the set.
- E. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is a linearly independent set of vectors unless $\mathbf{u}_4 = \mathbf{0}$.
- F. none of the above

II. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a linearly independent set of vectors.
Select the best statement.

- A. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is also a linearly independent set of vectors unless $\mathbf{u}_4 = \mathbf{0}$.
- B. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ could be a linearly independent or linearly dependent set of vectors depending on the vectors chosen.
- C. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is also a linearly independent set of vectors unless $\mathbf{u}_4$ is a scalar multiple another vector in the set.
- D. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is always a linearly independent set of vectors.
- E. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is always a linearly dependent set of vectors.
- F. none of the above

III. Let $\mathbf{u}_4$ be a linear combination of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.
Select the best statement.

- A. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is never a linearly dependent set of vectors.
- B. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is always a linearly independent set of vectors.
- C. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is never a linearly independent set of vectors.
- D. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ could be a linearly dependent or linearly dependent set of vectors depending on the vector space chosen.
- E. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a linearly dependent set of vectors unless one of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is the zero vector.
- F. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ could be a linearly dependent or linearly dependent set of vectors depending on the vectors chosen.
- G. none of the above

IV. Assume $\mathbf{u}_4$ is not a linear combination of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.
Select the best statement.

- A. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is always a linearly dependent set of vectors.
- B. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is a linearly independent set of vectors unless one of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is the zero vector.
- C. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ could be a linearly independent or linearly dependent set of vectors depending on the vector space chosen.
- D. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is a linearly dependent set precisely when $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a linearly dependent set.
- E. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is never a linearly dependent set of vectors.
- F. none of the above

V. Let $\mathbf{u} = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$.
We want to determine by inspection (with minimal computation) if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent or independent.
Choose the best answer.

- A. The set is linearly independent because we only have two vectors and they are not scalar multiples of each other.
- B. The set is linearly dependent because two of the vectors are the same.
- C. The set is linearly dependent because one of the vectors is a scalar multiple of another vector.
- D. The set is linearly dependent because one of the vectors is the zero vector.
- E. The set is linearly dependent because the number of vectors in the set is greater than the dimension of the vector space.
- F. We cannot easily tell if the set is linearly dependent or linearly independent.

[2] from WeBWorK