Dawson College: Linear Algebra (SCIENCE): 201-NYC-05-S2: Fall 2018

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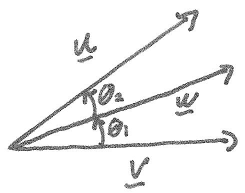
Quiz 9

This quiz is graded out of 13 marks. No books, watches, notes or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, and let $k = ||\vec{u}||$ and $l = ||\vec{v}||$. Prove that the vector $\vec{w} = l\vec{u} + k\vec{v}$ bisects the angle between $\vec{u}$ and $\vec{v}$.



$$\underline{u}\cdot\underline{w} = \|\underline{u}\|\|\underline{w}\|\cos\theta_2$$

$$\cos\theta_2 = \frac{\underline{u}\cdot\underline{w}}{\|\underline{u}\|\|\underline{w}\|}$$

$$\cos\theta_2 = \frac{\underline{u}\cdot(l\underline{u}+k\underline{v})}{\|\underline{u}\|\|\underline{w}\|}$$

$$= \frac{\underline{u}\cdot(l\underline{u})+\underline{u}(k\underline{v})}{\|\underline{u}\|\|\underline{w}\|}$$

$$= \frac{l(\underline{u}\cdot\underline{u})+k(\underline{u}\cdot\underline{v})}{\|\underline{u}\|\|\underline{w}\|}$$

$$= \frac{l\|\underline{u}\|^2+k\|\underline{u}\|\|\underline{v}\|\cos(\theta_1+\theta_2)}{\|\underline{u}\|\|\underline{w}\|}$$

$$= \frac{\|\underline{v}\|\|\underline{u}\|^2+\|\underline{u}\|\|\underline{u}\|\|\underline{v}\|\cos(\theta_1+\theta_2)}{\|\underline{u}\|\|\underline{w}\|}$$

$$= \frac{\|\underline{u}\|\|\underline{v}\|(1+\cos(\theta_1+\theta_2))}{\|\underline{w}\|}$$

$$\underline{v}\cdot\underline{w} = \|\underline{v}\|\|\underline{w}\|\cos\theta_1$$

$$\cos\theta_1 = \frac{\underline{v}\cdot\underline{w}}{\|\underline{v}\|\|\underline{w}\|}$$

$$\cos\theta_1 = \frac{\underline{v}\cdot(l\underline{u}+k\underline{v})}{\|\underline{v}\|\|\underline{w}\|}$$

$$= \frac{\underline{v}\cdot(l\underline{u})+\underline{v}(k\underline{v})}{\|\underline{v}\|\|\underline{w}\|}$$

$$= \frac{l(\underline{v}\cdot\underline{u})+k(\underline{v}\cdot\underline{v})}{\|\underline{v}\|\|\underline{w}\|}$$

$$= \frac{l\|\underline{v}\|\|\underline{u}\|\cos(\theta_1+\theta_2)+k\|\underline{v}\|^2}{\|\underline{v}\|\|\underline{w}\|}$$

$$= \frac{\|\underline{v}\|\|\underline{v}\|\|\underline{u}\|\cos(\theta_1+\theta_2)+\|\underline{u}\|\|\underline{v}\|^2}{\|\underline{v}\|\|\underline{w}\|}$$

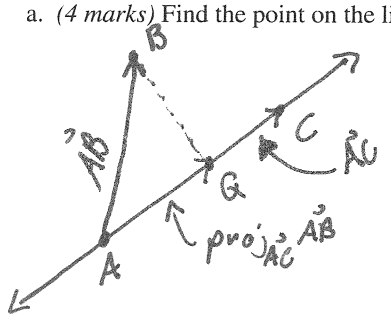
$$= \frac{\|\underline{u}\|\|\underline{v}\|(1+\cos(\theta_1+\theta_2))}{\|\underline{w}\|}$$

$\therefore \cos\theta_1 = \cos\theta_2$

$\therefore$ $\underline{w}$ bisects the angle between $\underline{u}$ and $\underline{v}$.

Question 2.[1] Given the following points: $A = (3,1,1,1)$, $B = (2,1,3,0)$, and $C = (1,0,3,1)$.

a. *(4 marks)* Find the point on the line containing A and C that is closest to B.



$$\vec{AB} = B - A = (2,1,3,0)-(3,1,1,1) = (-1,0,2,-1)$$

$$\vec{AC} = C - A = (1,0,3,1)-(3,1,1,1) = (-2,-1,2,0)$$

$$\vec{AQ} = \mathrm{proj}_{\vec{AC}}\vec{AB}$$

$$\vec{AQ} = \frac{\vec{AB}\cdot\vec{AC}}{\vec{AC}\cdot\vec{AC}}\vec{AC}$$

$$= \frac{(-1)(-2)+(0)(-1)+2(2)+(-1)(0)}{(-2)(-2)+(-1)(-1)+2(2)+(0)(0)}(-2,-1,2,0)$$

$$= \frac{6}{9}(-2,-1,2,0)$$

$$= \frac{2}{3}(-2,-1,2,0)$$

$$Q - A = \mathrm{proj}_{\vec{AC}}\vec{AB}$$

$$Q = A + \mathrm{proj}_{\vec{AC}}\vec{AB}$$

$$Q = (3,1,1,1)+\tfrac{2}{3}(-2,-1,2,0)$$

$$= \left(\tfrac{5}{3},\tfrac{1}{3},\tfrac{7}{3},1\right)$$

b. *(2 marks)* Find the area of the triangle with vertices at points A, B, and C.

$$\text{Area} = \frac{\|\vec{QB}\|\|\vec{AC}\|}{2} = \frac{\sqrt{(\tfrac{1}{3})^2+(\tfrac{2}{3})^2+(\tfrac{2}{3})^2+(-1)^2}\sqrt{(-2)^2+(-1)^2+2^2+0^2}}{2}$$

$$\vec{QB} = B - Q = (2,1,3,0)-\left(\tfrac{5}{3},\tfrac{1}{3},\tfrac{7}{3},1\right) = \left(\tfrac{1}{3},\tfrac{2}{3},\tfrac{2}{3},-1\right)$$

$$= \frac{\sqrt{\tfrac{18}{9}}\sqrt{9}}{2} = \frac{3\sqrt{2}}{2}$$

Question 3. *(2 marks)* Determine whether the following statement is true or false. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

If the relationship $\mathrm{proj}_{\vec{a}}\vec{u} = \mathrm{proj}_{\vec{a}}\vec{v}$ holds some nonzero vector $\vec{a}$, then $\vec{u} = \vec{v}$.

False, if $\underline{a} = (1,0)$ and $\underline{u} = (1,1)$, $\underline{v} = (1,2)$ then $\mathrm{proj}_{\underline{a}}\underline{u} = \mathrm{proj}_{\underline{a}}\underline{v} = (1,0)$ but $\underline{u} \neq \underline{v}$.

[1] modified from a John Abbott Final Examination