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Quiz 12

This quiz is graded out of 12 marks. No books, watches, notes or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1.[1] *(10 marks)* Let $V = \{x \mid x \in \mathbb{R} \text{ and } x > 3\}$. For $\mathbf{u}, \mathbf{v} \in V$ and $a \in \mathbb{R}$ vector addition is defined as $\mathbf{u} +_V \mathbf{v} = \mathbf{uv} - 3(\mathbf{u} + \mathbf{v}) + 12$ and scalar multiplication is defined as $a \cdot_V \mathbf{u} = (\mathbf{u} - 3)^a + 3$. Show that V with the above two operations is a vector space over $\mathbb{R}$. Let $u, v, w \in V$ and $r, s \in \mathbb{R}$

① Closure under addition:

$u+v = uv - 3(u+v) + 12 \in V$ since $uv - 3(u+v) + 12 = uv - 3u - 3v + 12 = \underbrace{v}_{>3}(\underbrace{u-3}_{>0}) + 3(\cancel{4} - \underbrace{\cancel{u}}_{>1}) > 3$

② Commutativity of vector addition: $u+v = uv - 3(u+v) + 12 = vu - 3(v+u) + 12 = v+u$

③ associativity of vector addition: $u+(v+w) = u +_V (vw - 3(v+w) + 12) = u(vw - 3(v+w) + 12)$
$-3(u + vw - 3(v+w) + 12) + 12$
$= uvw - 3uv - 3uw + 12u$
$-3u - 3vw + 9v + 9w - 36 + 12$
$= uvw + 9u + 9v + 9w - 3uv$
$-3vw - 3uw - 24$

$(u+v)+w = (uv - 3(u+v) + 12) +_V w = (uv - 3(u+v) + 12)w$
$-3((uv - 3(u+v) + 12) + w) + 12$
$= uvw - 3uw - 3vw + 12w - 3uv$
$+ 9u + 9v - 36 - 3w + 12$
$= uvw + 9u + 9v + 9w - 3uw - 3vw$
$-3uv - 24$

④ zero vector: Let $0 = z$

$v + 0 = v$

$v = v + z$

$v = vz - 3(v+z) + 12$

$v = vz - 3v - 3z + 12$

$4v - 12 = vz - 3z$

$4(v-3) = z(v-3)$

$4 = z$ $\quad \therefore 0 = 4 \in V$

⑤ Every vector has an additive inverse: $v + x = 0$

$(vx - 3(v+x) + 12) = 4$

$vx - 3x = -8 + 3v$

$x = \dfrac{-8+3v}{v-3} = \dfrac{1 - 1 - 8 + 3v}{v-3}$

$= \dfrac{1}{v-3} + 3\dfrac{\cancel{(v-3)}}{\cancel{(v-3)}}$

$= \underbrace{\dfrac{1}{v-3}}_{>0} + 3 > 3$

$\therefore x \in V$

⑥ Closure under scalar multiplication.

$r \cdot u = \underbrace{(u-3)^r}_{>0} + 3 > 3 \quad \therefore r \cdot u \in V$

⑦ distributivity over scalar addition:

$(r+s)u = ru + su$

LHS $= (r+s)u = (u-3)^{r+s} + 3$

RHS $= ru +_V su = (u-3)^r + 3 +_V (u-3)^s + 3 = ((u-3)^r + 3)((u-3)^s + 3) - 3((u-3)^r + 3 + (u-3)^s + 3) + 12$
$= (u-3)^r(u-3)^s + 3(u-3)^s + 3(u-3)^r + 9 - 3(u-3)^r - 9$
$- 3(u-3)^s - 9 + 12$
$= (u-3)^{r+s} + 3$

⑧ scalar distributivity over vector addition: $r(u+v) = ru + rv$

LHS $= r(u+v) = r(uv - 3(u+v) + 12)$
$= (uv - 3(u+v) + 12 - 3)^r + 3$
$= (uv - 3(u+v) + 9)^r + 3$

RHS $= ru + rv$
$= (u-3)^r + 3 +_V (v-3)^r + 3$
$= ((u-3)^r + 3)((v-3)^r + 3) - 3((u-3)^r + 3 + (v-3)^r + 3) + 12$
$= (u-3)^r(v-3)^r + \cancel{3(u-3)^r} + \cancel{3(v-3)^r} + 9 - \cancel{3(u-3)^r} - 9 - \cancel{3(v-3)^r} - 9 + 12 = (u-3)^r(v-3)^r + 3 = ((u-3)(v-3))^r + 3 = $ LHS

⑨ associativity of scalar multiplication: $(rs)u = r(su)$

LHS $= (rs)u = (u-3)^{rs} + 3$

RHS $= r(su) = r((u-3)^s + 3) = ((u-3)^s + 3 - 3)^r + 3 = $ LHS

⑩ $1 \cdot u = (u-3)^1 + 3 = u$.

Question 2. *(2 marks)* Determine whether the set equipped with the given operations is a vector space. For those that are not vector spaces identify and show an axiom that fails.

The set of all triples of real numbers with the standard vector addition but with scalar multiplication defined by $k(x, y, z) = (k^2x, k^2y, k^2z)$

Not a vector space since it fails distributivity over scalar multiplication:

$(r+s) \cdot v \neq rv + sv$. Since

$(r+s) \cdot v = (r+s) \cdot (x, y, z) = ((r+s)^2x, (r+s)^2y, (r+s)^2z) = ((r^2 + 2rs + s^2)x, (r^2 + 2rs + s^2)y, (r^2 + 2rs + s^2)z)$

$r \cdot v + s \cdot v = r \cdot (x, y, z) + s \cdot (x, y, z) = (r^2x, r^2y, r^2z) + (s^2x, s^2y, s^2z) = ((r^2 + s^2)x, (r^2 + s^2)y, (r^2 + s^2)z)$

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[1] from WeBWorK