Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S2: Fall 2018

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Quiz 15

This quiz is graded out of 16 marks. No books, watches, notes or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(1 mark each)* Fill in the blank.

a. The vector space of all diagonal $n \times n$ matrices has dimension $\underline{n}$.

b. The vector space of all symmetric $n \times n$ matrices has dimension $\underline{\frac{n(n+1)}{2}}$.

Question 2.[1] A matrix A and its reduced row echelon form U are given

$$A = \begin{bmatrix} 1 & 4 & -2 & 4 & v & 3 & 6 \\ 3 & 12 & -6 & 12 & w & 2 & 15 \\ -2 & -8 & 4 & -8 & x & -1 & -13 \\ 1 & 4 & -2 & 5 & y & 0 & 3 \\ 3 & 12 & -6 & 12 & z & 3 & 10 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 4 & -2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\mathbf{a}_i$ denote the ith column of A, and $\mathbf{u}_j$ denote the jth column of U, so that

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & \mathbf{a}_7 \end{bmatrix} \text{ and } U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 & \mathbf{u}_5 & \mathbf{u}_6 & \mathbf{u}_7 \end{bmatrix}$$

and let $V = \text{span}(\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6, \mathbf{a}_7\})$. You may use the above notation in your answers to the following questions. *Justify completely!!!*

a. *(2 marks)* Is $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ a basis for V. Not a basis for V since the set is linearly dependent because $4\underline{a_1} = \underline{a_2}$.

b. *(4 marks)* Find a basis for V.

c. *(2 marks)* Express $\mathbf{a}_7$ with respect to the basis of V found in part b.

d. *(2 marks)* Find the values of the entries v, w, x, y, and z in the matrix A.

e. *(4 marks)* Find a basis for $\{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}$.

b) To find a basis of V we need to remove from the set $\{\underline{a_1}, \underline{a_2}, \dots, \underline{a_7}\}$ the vectors that can be expressed as a linear combination of the others (or that we have the largest linearly independent subset). So we remove $\underline{a_2}, \underline{a_3}$ since they are multiples of $\underline{a_1}$. We also remove $\underline{a_7}$ since it can be written as a linear combination of $\underline{a_1}$, $\underline{a_5}, \underline{a_6}$. Let $c_1\underline{a_1} + \dots + c_7\underline{a_7} = \underline{0}$, we obtain the following homogeneous system $[A \mid 0] \sim$ Gauss Jordan $\sim [U \mid 0]$, its solution set $c_7 = s$, $c_3 = t$, $c_2 = r$ $(s,t,r \in \mathbb{R})$.

$(c_1, c_2, c_3, c_4, c_5, c_6, c_7) = (-4r+2t-s, r, t, 0, 2s, -3s, s)$ Let $s = 1$, $t = r = 0$ then $(c_1, c_2, c_3, c_4, c_5, c_6, c_7) = (-1, 0, 0, 0, 2, -3, 1)$. Hence

$-\underline{a_1} + 2\underline{a_5} - 3\underline{a_6} + \underline{a_7}$

$\underline{a_7} = \underline{a_1} - 2\underline{a_5} + 3\underline{a_6}$ $*$

$\therefore \text{span}\{\underline{a_1}, \underline{a_4}, \underline{a_5}, \underline{a_6}\} = V$ and it follows that $\{\underline{a_1}, \underline{a_4}, \underline{a_5}, \underline{a_6}\}$ is linearly independent since $c_1\underline{a_1} + c_2\underline{a_4} + c_3\underline{a_5} + c_4\underline{a_6} = \underline{0}$ has only the trivial solution. Because $[\underline{a_1}\ \underline{a_4}\ \underline{a_5}\ \underline{a_6} \mid 0] \sim$ Gauss Jordan $\sim [\underline{e_1}\ \underline{e_2}\ \underline{e_3}\ \underline{e_4} \mid 0]$ by A and U. $\therefore B = \{\underline{a_1}, \underline{a_4}, \underline{a_5}, \underline{a_6}\}$

c) $(\underline{a_7})_B = (1, 0, -2, 3)$ by $*$

d) By $*$

$\underline{a_7} = \underline{a_1} - 2\underline{a_5} + 3\underline{a_6}$

$\underline{a_5} = \frac{1}{2}\underline{a_1} + \frac{3}{2}\underline{a_6} - \frac{1}{2}\underline{a_7}$

$$\begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} = \underline{a_5} = \frac{1}{2}\begin{pmatrix} 1 \\ 3 \\ -2 \\ 1 \\ 3 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} 3 \\ 2 \\ -1 \\ 0 \\ 3 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 6 \\ 15 \\ -13 \\ 3 \\ 10 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 4 \\ -1 \\ 1 \end{pmatrix}$$

e) $[A \mid 0] \sim$ Gauss Jordan $\sim [U \mid 0]$

Let $x_7 = s$, $x_3 = t$, $x_2 = r$ $(r, s, t \in \mathbb{R})$

$\therefore (x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (-4r+2t-s, r, t, 0, 2s, -3s, s)$

$= r\underbrace{(-4, 1, 0, 0, 0, 0, 0)}_{\underline{v_1}} + t\overbrace{(2, 0, 1, 0, 0, 0, 0)}^{\underline{v_2}} + s\underbrace{(-1, 0, 0, 0, 2, -3, 1)}_{\underline{v_3}}$

$\therefore \text{span}\{\underline{v_1}, \underline{v_2}, \underline{v_3}\} = \{x \mid Ax = 0\}$

By observation it is clear that $c_1\underline{v_1} + c_2\underline{v_2} + c_3\underline{v_3} = \underline{0}$ has only the trivial solution.

$\therefore \{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ is a basis of $\{x \mid Ax = 0\}$

[1] modified from John Abbott Final examination