Dawson College: Fall 2019: Linear Algebra (SCIENCE): 201-NYC-05-S5: **Quiz 5** name: Y. Lamontagne

No books, watches, notes or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1.[1] *(3 marks)* Show that if A and B are square matrices such that AB is invertible, then A can be written as a product of elementary matrices.

Since AB is invertible we have $AB(AB)^{-1} = I$. Let $C = B(AB)^{-1}$ then by a theorem seen in class A is invertible and $A^{-1} = B(AB)^{-1}$. Since A is invertible it can be expressed as a product of elementary matrices by the equivalence theorem.

Question 2.[1] Given the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 5 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

a. *(3 marks)* Find A^{-1}.

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \underset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \underset{\frac{1}{5}R_1 \to R_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1/5 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{-R_2+R_3 \to R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1/5 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \quad \therefore A^{-1} = \begin{bmatrix} 0 & 1/5 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

b. *(2 marks)* Express A^{-1} as a product of elementary matrices, if possible. **Justify your work!**

$$E_3E_2E_1A = I$$
$$E_3E_2E_1AA^{-1} = IA^{-1}$$
$$E_3E_2E_1 = A^{-1}$$

where $I_3 \underset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$

$I_3 \underset{\frac{1}{5}R_1 \to R_1}{\sim} \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_2$

$I_3 \underset{-R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = E_3$

c. *(2 marks)* Express A as a product of elementary matrices, if possible. **Justify your work!**

Since $A^{-1} = E_3E_2E_1$
$$(A^{-1})^{-1} = (E_3E_2E_1)^{-1}$$
$$A = E_1^{-1}E_2^{-1}E_3^{-1}$$

note that the inverse of elementary matrices are elementary matrices.

where $I_3 \underset{R_1 \leftrightarrow R_1}{\sim} = E_1 = E_1^{-1}$

$I_3 \underset{5R_1 \to R_1}{\sim} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_2^{-1}$

$I_3 \underset{R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = E_3^{-1}$

[1] From a past Dawson final examination

Question 3. Determine whether the following statements are true or false. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

a. *(3 marks)* If A and B are row equivalent matrices, then the linear systems $Ax = 0$ and $Bx = 0$ have the same solution set.

True, since A and B are row equivalent $A \sim k$ elem row op $\sim B$.
Suppose we apply Gauss-Jordan on the augmented matrix $[B|0]$
we have $[B|0] \sim l$ elem. row. op $\sim [R|0]$. and it follows that

$[A|0] \sim k$ elem row op $\sim [B|0] \sim l$ elem row op $\sim [R|0]$

Hence $Ax=0$ and $Bx=0$ since they have the same reduced augmented matrices.

b. *(3 marks)* Let A be an $n \times n$ matrix and S is an $n \times n$ invertible matrix. If $\cancel{\mathbf{x}}$ is a solution to the linear system $(S^{-1}AS)\mathbf{x} = \mathbf{b}$, then $S\mathbf{x}$ is a solution to the linear system $A\mathbf{y} = S\mathbf{b}$. True,

Since $\underline{x}$ is a solution of $S^{-1}ASx = b$ it satisfies the system. So

$S^{-1}AS\underline{x} = b$

$SS^{-1}AS\underline{x} = Sb$

$IAS\underline{x} = Sb$

$AS\underline{x} = Sb$

$\therefore$ $S\underline{x}$ satisfies the system $Ay = Sb$

c. *(3 marks)* Every elementary matrix is invertible.

True, Suppose E is an elementary matrix, it is obtained by performing a single elementary row operation on I. If we perform Gauss-Jordan on E we obtain its RREF which is I since Gauss-Jordan would consist of performing the inverse row op. on E. Hence E is invertible by the equivalence thm.

Bonus. *(3 marks)* Find the formula for the n-th power of this matrix. $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$