Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S7: Fall 2022: **Quiz 12** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. Consider the set

$$V = \{(x,\, y) \mid x + y = 2\}$$

under the following operations:

$$(x_1,\, y_1) + (x_2,\, y_2) = (x_1 \cdot x_2,\, y_1 + y_2 - 1) \qquad k \cdot (x,\, y) = (kx - k + 1,\, y^k)$$

a. *(1 mark)* Find $2 \cdot (3, -1)$.

$2 \cdot (3,-1) = (2(3)-2+1, (-1)^2) = (5, 1)$

b. *(3 marks)* Show that V contains a zero vector (in the sense of a vector space).

Let $\underline{0} = (a, b)$ and $\underline{v} = (x, y) \in V$

$\underline{v} + \underline{0} = \underline{v}$

$(x, y) + (a, b) = (x, y)$

$(xa, y+b-1) = (x, y)$

$xa = x \Rightarrow a = 1$

$y+b-1 = y \Rightarrow b = 1$

$\therefore\ \underline{0} = (1,1) \in V$ since $1+1 = 2$

c. *(3 marks)* Does V contain an additive inverse for $\vec{v} = (-1, 3)$? If so, find it. Justify.

$\underline{v} + \underline{w} = \underline{0}$

$(-1, 3) + (x, y) = (1, 1)$

$(-1 \cdot x, 3 + y - 1) = (1, 1)$

$-x = 1 \Rightarrow x = -1$

$3+y-1 = 1 \Rightarrow y = -1$

$\underline{w} = (-1,-1) \notin V$ since $(-1)+(-1) \neq 2$.

d. *(2 marks)* Is V with the given operations a vector space? Justify.

V is not a vector space since axiom 5 fails because it does not contain $\underline{v} = (-1,3) \in V$ does not have an additive inverse in V.

Question 2. *(3 marks)* If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

There does not exist a vector space with two elements.

True,

Suppose that $V = \{\underline{a}, \underline{b}\}$ is a vector space. Since V is a vector space then $\underline{0} \in V$ (WOLOG $\underline{0} = \underline{a}$). Then $\underline{b} \neq \underline{0}$ and $\underline{b} + \underline{a} = \underline{b}$ since $\underline{a} = \underline{0}$ $\therefore$ $\underline{a}$ is not the additive inverse of $\underline{b}$. And $\underline{b} + \underline{b} = 2\underline{b} \neq \underline{0}$ by thm 1.1 since $2 \neq 0$ and $\underline{b} \neq \underline{0}$.

$\therefore$ $\underline{b}$ does not have an additive inverse.

Bonus Question.

definition:[1] A *group* G is a set of elements satisfying the four conditions below, relative to some binary operation.

1. $\exists e \in G$ s.t. $\forall g \in G : e \cdot g = g \cdot e = g$. (Identity.)
2. $\forall x, y, z \in G : (x \cdot y) \cdot z = x \cdot (y \cdot z)$. (Associativity.)
3. $\forall x \in G, \exists y \in G$ s.t. $x \cdot y = y \cdot x = e$. (Inverse.)
4. $\forall x, y \in G : x \cdot y \in G$. (Closure.)

a. *(2 marks)* Show that the identity is unique.

b. *(2 marks)* Is a vector space a group?

[1] https://web.williams.edu/Mathematics/sjmiller/public_html/mathlab/public_html/handouts/GroupTheoryIntro.tex