Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S7: Fall 2022: **Quiz 13** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(5 marks)* Determine whether the following is a subspace of $\mathscr{M}_{n\times n}$: The set of all $n\times n$ matrices such that $AB = BA$ for some fixed $n\times n$ matrix B.

Let $W = \{A \mid A \in M_{n\times n} \text{ and } AB = BA\}$ where $B \in M_{n\times n}$

① Closure under addition:

Let $A_1, A_2 \in W \Rightarrow A_1B = BA_1$ and $A_2B = BA_2$ $(*)$

$A_1 + A_2 \in W$ since
$$\begin{aligned}(A_1+A_2)B &= A_1B + A_2B \\ &= BA_1 + BA_2 \quad \text{by } (*) \\ &= B(A_1+A_2)\end{aligned}$$

$\therefore$ closed under addition.

② Closure under scalar multiplication

Let $k \in \mathbb{R}$ and $A \in W \Rightarrow AB = BA$ $(**)$

$kA \in W$ since
$$\begin{aligned}(kA)B &= kAB \\ &= kBA \quad \text{by } (**) \\ &= B(kA)\end{aligned}$$

$\therefore$ closed under scalar multiplication.

$\therefore$ W is a subspace by the subspace test.

Question 2. *(5 marks)* Let V be the solution space of the equation $4x - y + 2z = 0$, and let W be the subspace of $\mathbb{R}^3$ spanned by $(1,1,1)$. Find a vector $\vec{v}$ in V and a vector $\vec{w}$ in W for which $\vec{v} + \vec{w} = (1,0,1)$.

Lets determine which vectors span V. Let $y = s$, $z = t$

$$\begin{aligned}4x - s + 2t &= 0 \\ 4x &= s - 2t \\ x &= \tfrac{1}{4}s - \tfrac{1}{2}t\end{aligned}$$

$$\therefore (x,y,z) = \left(\tfrac{1}{4}s - \tfrac{1}{2}t, s, t\right) = \underbrace{s\left(\tfrac{1}{4}, 1, 0\right) + t\left(-\tfrac{1}{2}, 0, 1\right)}_{\in V}$$

$\therefore V = \text{span}(\{(1,4,0), (1,0,-2)\})$

So lets solve $\underbrace{c_1(1,4,0) + c_2(1,0,-2)}_{\in V} + \underbrace{c_3(1,1,1)}_{\in W} = (1,0,1)$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 0 & 1 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix}
\underset{-4R_1+R_2\to R_2}{\sim}
\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & -3 & -4 \\ 0 & -2 & 1 & 1 \end{bmatrix}
\underset{R_2 \leftrightarrow R_3}{\sim}
\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & -4 & -3 & -4 \end{bmatrix}
\underset{-2R_2+R_3\to R_3}{\sim}
\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -5 & -6 \end{bmatrix}$$

$$\underset{\substack{5R_1\to R_1 \\ 5R_2\to R_2}}{\sim}
\begin{bmatrix} 5 & 5 & 5 & 5 \\ 0 & -10 & 5 & 5 \\ 0 & 0 & -5 & -6 \end{bmatrix}
\underset{\substack{R_3+R_1\to R_1 \\ R_3+R_2\to R_2}}{\sim}
\begin{bmatrix} 5 & 5 & 0 & -1 \\ 0 & -10 & 0 & -1 \\ 0 & 0 & -5 & -6 \end{bmatrix}
\underset{2R_1\to R_1}{\sim}
\begin{bmatrix} 10 & 10 & 0 & -2 \\ 0 & -10 & 0 & -1 \\ 0 & 0 & -5 & -6 \end{bmatrix}
\underset{R_2+R_1\to R_1}{\sim}
\begin{bmatrix} 10 & 0 & 0 & -3 \\ 0 & -10 & 0 & -1 \\ 0 & 0 & -5 & -6 \end{bmatrix}$$

$$\underset{\substack{\frac{1}{10}R_1\to R_1 \\ -\frac{1}{10}R_2\to R_2 \\ -\frac{1}{5}R_3\to R_3}}{\sim}
\begin{bmatrix} 1 & 0 & 0 & -3/10 \\ 0 & 1 & 0 & 1/10 \\ 0 & 0 & 1 & 6/5 \end{bmatrix}$$

So $\underbrace{-\tfrac{3}{10}(1,4,0) + \tfrac{1}{10}(1,0,-2)}_{\in V} + \underbrace{\tfrac{6}{5}(1,1,1)}_{\in W} = (1,0,1)$