Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S5: Fall 2024: **Quiz 1** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(3 marks)* Determine whether the following statement is true or false. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

If AB and BA are both defined, then AB and BA are square matrices.

True. Let A be $m\times n$ and B be $p\times q$ matrices. Since AB is defined it implies that # col of A = # rows of B $\therefore n=p$

Since BA is defined it implies that # col of B = # rows of A $\therefore q=m$

$\therefore A_{m\times n}B_{p\times q}$ is a $m\times q$ matrix but since $q=m$, it is a $m\times m$ matrix.

Similarly BA is an $n\times n$ matrix.

Question 2. Find the values of k for which the following system has:

$$\begin{cases} x + y + 2z = -1 \\ -x+(k^2-2)y+(2k-4)z = k+4 \\ x + y+(k^2+k)z = k+1 \end{cases}$$

a. Exactly one solution, justify.

b. No solutions, justify.

c. Infinitely many solutions, justify.

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ -1 & k^2-2 & 2k-4 & k+4 \\ 1 & 1 & k^2+k & k+1 \end{bmatrix} \sim \begin{matrix} R_1+R_2\to R_2 \\ -R_1+R_3\to R_3 \end{matrix} \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & k^2-1 & 2k-2 & k+3 \\ 0 & 0 & k^2+k-2 & k+2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & (k-1)(k+1) & 2(k-1) & k+3 \\ 0 & 0 & (k+2)(k-1) & k+2 \end{bmatrix}$$

If $k=-2$ then $\begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 3 & -6 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and ∞-many solutions since #leading entries in var. col. < #var

If $k=1$ then $\begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ and no solutions since leading entry in constant column.

If $k=-1$ then $\begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & -2 & 1 \end{bmatrix} \sim_{-2R_2+R_3\to R_3} \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

and ∞-many solutions since # leading entries in var. col. < #var.

If $k\neq \pm1,-2$ then unique solution since #leading entries in var. col. = #var.

Question 3. *(3 marks)* Determine whether the following statement is true or false. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

If A and B are square matrices of the same order, then $\mathrm{tr}(AB) = \mathrm{tr}(A)\mathrm{tr}(B)$.

False, If $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 \\ 0 & 4 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 5 & 12 \end{bmatrix}$, $tr(AB) = 10 + 12 = 22$

but $tr(A)tr(B) = (2+3)(5+4) = 5(9) = 45 \neq tr(AB)$

Question 4. *(6 marks)* Find **all** matrices A, if any, such that A and B commute where $B = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$AB = BA$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} 2a+5b & 3a+4b \\ 2c+5d & 3c+4d \end{bmatrix} = \begin{bmatrix} 2a+3c & 2b+3d \\ 5a+4c & 5b+4d \end{bmatrix}$$

$$\left.\begin{aligned} 2a+5b &= 2a+3c \\ 3a+4b &= 2b+3d \\ 2c+5d &= 5a+4c \\ 3c+4d &= 5b+4d \end{aligned}\right\} \quad \begin{aligned} 5b - 3c &= 0 \quad (1) \\ 3a + 2b - 3d &= 0 \quad (2) \\ -5a - 2c + 5d &= 0 \quad (3) \\ -5b + 3c &= 0 \quad (4) \end{aligned}$$

(1) and (4) are multiples, no need for (4) when solving

$$\begin{bmatrix} 3 & 2 & 0 & -3 & 0 \\ -5 & 0 & -2 & 5 & 0 \\ 0 & 5 & -3 & 0 & 0 \end{bmatrix}$$

$\sim$ $3R_2 \to R_2$ $\begin{bmatrix} 3 & 2 & 0 & -3 & 0 \\ -15 & 0 & -6 & 15 & 0 \\ 0 & 5 & -3 & 0 & 0 \end{bmatrix}$

$\sim$ $5R_1 + R_2 \to R_2$ $\begin{bmatrix} 3 & 2 & 0 & -3 & 0 \\ 0 & 10 & -6 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 \end{bmatrix}$

$\sim$ $\frac{1}{2}R_2 \to R_2$ $\begin{bmatrix} 3 & 2 & 0 & -3 & 0 \\ 0 & 5 & -3 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 \end{bmatrix}$

$\sim$ $-R_2 + R_3 \to R_3$ $\begin{bmatrix} 3 & 2 & 0 & -3 & 0 \\ 0 & 5 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\sim$ $5R_1 \to R_1$ $\begin{bmatrix} 15 & 10 & 0 & -15 & 0 \\ 0 & 5 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\sim$ $-2R_2 + R_1 \to R_1$ $\begin{bmatrix} 15 & 0 & 6 & -15 & 0 \\ 0 & 5 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\sim$ $\frac{1}{15}R_1 \to R_1$, $\frac{1}{5}R_2 \to R_2$ $\begin{bmatrix} 1 & 0 & \frac{2}{5} & -1 & 0 \\ 0 & 1 & -3/5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Let $c = s$, $d = t$ $\quad s,t \in \mathbb{R}$

$a = -\frac{2}{5}s + t$

$b = \frac{3}{5}s$

$\therefore A = \begin{bmatrix} -\frac{2}{5}s + t & \frac{3}{5}s \\ s & t \end{bmatrix}$ $\quad s,t \in \mathbb{R}$.