Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S5: Fall 2024: **Quiz 4** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1.

a. *(3 marks)* Determine all s and t such that the given matrix is symmetric

$$A = \begin{bmatrix} s & 2s & st \\ t & -1 & s \\ t & s^2 & s \end{bmatrix} = \begin{bmatrix} s & t & t \\ 2s & -1 & s^2 \\ st & s & s \end{bmatrix} = A^T$$

$\Rightarrow$ ① $2s = t$

② $st = t$

③ $s = s^2 \Rightarrow s^2 - s = 0$

$s(s-1) = 0$

$s = 0$ $s = 1$

sub $s=0$ in ①

$2(0) = t$

$t = 0$

satisfies ②

$\therefore (s,t) = (0,0)$ is a solution

sub $s=1$ in ①

$2(1) = t$

$t = 2$

satisfies ②

$\therefore (s,t) = (1,2)$ is also a solution.

b. *(2 marks)* If A and B are invertible symmetric matrices such that A and B commute show that $A^{-1}B^{-1}$ is also invertible and symmetric.

premise:
A and B are invertible symmetric matrices

conclusion:
$A^{-1}B^{-1}$ is invertible and symmetric.

The product of invertible matrices is invertible
$\therefore A^{-1}B^{-1}$ is invertible

For symmetric, we need to show $(A^{-1}B^{-1})^T = (A^{-1}B^{-1})$

LHS $= (A^{-1}B^{-1})^T$

$= ((BA)^{-1})^T$ since A and B are inv.

$= ((AB)^{-1})^T$ since A and B commute

$= ((AB)^T)^{-1}$

$= (B^TA^T)^{-1}$

$= (BA)^{-1}$ since A and B are symmetric

$= A^{-1}B^{-1}$

$=$ RHS

Question 2. *(5 marks)* Solve for x where

$$\begin{vmatrix} \sin x & \cos x \\ -\cos x & \sin x \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ e^x & 1 & e^x \\ 1 & e^x & 0 \end{vmatrix}$$

$\sin^2 x + \cos^2 x = 1C_{31} + e^x C_{32} + 0C_{33}$

$1 = 1 \cdot (-1)^{3+1} \begin{vmatrix} 1 & 1 \\ 1 & e^x \end{vmatrix} + e^x \begin{vmatrix} 1 & 1 \\ e^x & e^x \end{vmatrix}$

$1 = e^x - 1$

$2 = e^x$

$\ln 2 = \ln e^x$

$\ln 2 = x$

Question 3. *(3 marks)* Determine whether the following statement is true or false. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

Let $A\mathbf{x} = \mathbf{b}$ be any consistent system of linear equations, and let $\mathbf{x}_1$ be a fixed solution. Then every solution to the system can be written in the form $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_0$, where $\mathbf{x}_0$ is a solution to $A\mathbf{x} = \mathbf{0}$.

True,

Let $\underline{x}$ be any solution of $A\underline{x} = \underline{b}$. Then we can rewrite $\underline{x}$ as

$\underline{x} = \underline{x}_1 + (\underline{x} - \underline{x}_1)$

where $\underline{x}_0 = \underline{x} - \underline{x}_1$ and is a solution of $A\underline{x} = \underline{0}$ since
$$\begin{aligned} A\underline{x}_0 &= A(\underline{x} - \underline{x}_1) \\ &= A\underline{x} - A\underline{x}_1 \\ &= \underline{b} - \underline{b} \\ &= \underline{0} \end{aligned}$$

Question 4. *(5 marks)* Let A be a 3×3 matrix such that $\det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2$. Find the following: $\begin{vmatrix} a+b & 3d+3e & g+h \\ 2a-3b & 6d-9e & 2g-3h \\ 2c & 6f & 2i \end{vmatrix}$.

$$= \begin{matrix} -2R_1 + R_2 \to R_2 \\ \frac{1}{2}R_3 \to R_3 \end{matrix} \quad 2\begin{vmatrix} a+b & 3d+3e & g+h \\ -5b & -15e & -5h \\ c & 3f & i \end{vmatrix}$$

$$= \begin{matrix} \frac{1}{5}R_2 + R_1 \to R_1 \\ -\frac{1}{5}R_2 \to R_2 \end{matrix} \; (-5)(2) \begin{vmatrix} a & 3d & g \\ b & 3e & h \\ c & 3f & i \end{vmatrix}$$

$\frac{1}{3}C_2 \to C_2$

$$= (3)(-5)(2) \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

$$= -30 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$= -30(2)$

$= -60$

Bonus. *(3 marks)* Let A and B be $m\times n$ and $n\times m$ matrices, respectively. If $m > n$, show that AB is not invertible.