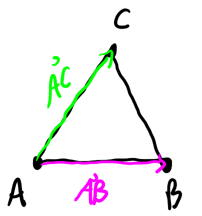
Dawson College: **Linear Algebra** (Science): 201-NYC-05-S5: Fall 2024: **Quiz 7** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. Consider the points $A(2,-2,4)$, $B(4,-1,1)$, and $C(3,-1,2)$.

a. *(3 marks)* Find the area of the triangle ABC.



$$\text{Area} = \frac{1}{2}\|\vec{AB}\times\vec{AC}\| = \frac{1}{2}\|(1,1,1)\| = \frac{1}{2}\sqrt{3} = \frac{\sqrt{3}}{2}$$

$\vec{AB} = \vec{OB} - \vec{OA} = (4,-1,1) - (2,-2,4) = (2,1,-3)$

$\vec{AC} = \vec{OC} - \vec{OA} = (3,-1,2) - (2,-2,4) = (1,1,-2)$

$\vec{AB}\times\vec{AC} = \left(\begin{vmatrix} 1 & 1 \\ -3 & -2 \end{vmatrix}, -\begin{vmatrix} 2 & 1 \\ -3 & -2 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \right) = (1,1,1)$

$\begin{matrix} 2 & 1 \\ 1 & 1 \\ -3 & -2 \end{matrix}$

b. *(3 marks)* Find the exact value of the tangent of the angle at the vertex A of the triangle ABC.

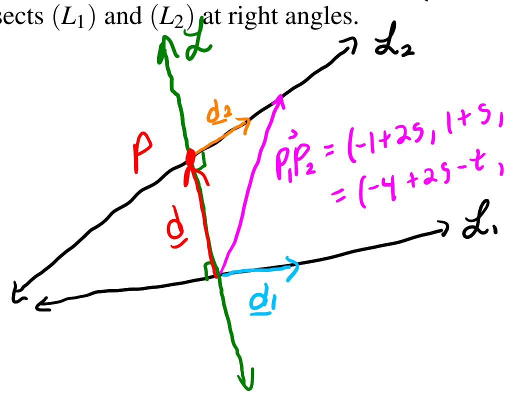
$$\tan A = \frac{\sin A}{\cos A} = \frac{\frac{\|\vec{AB}\times\vec{AC}\|}{\|\vec{AB}\|\|\vec{AC}\|}}{\frac{\vec{AB}\cdot\vec{AC}}{\|\vec{AB}\|\|\vec{AC}\|}} = \frac{\|\vec{AB}\times\vec{AC}\|}{\vec{AB}\cdot\vec{AC}} = \frac{\sqrt{3}}{9}$$

$\|\vec{AB}\times\vec{AC}\| = \|\vec{AB}\|\|\vec{AC}\|\sin\theta \Rightarrow \sin\theta = \dfrac{\|\vec{AB}\times\vec{AC}\|}{\|\vec{AB}\|\|\vec{AC}\|}$

$\vec{AB}\cdot\vec{AC} = \|\vec{AB}\|\|\vec{AC}\|\cos\theta \Rightarrow \cos\theta = \dfrac{\vec{AB}\cdot\vec{AC}}{\|\vec{AB}\|\|\vec{AC}\|}$

$\vec{AB}\cdot\vec{AC} = (2,1,-3)\cdot(1,1,-2) = 2+1+6 = 9$

Question 2. *(5 marks)* Given that the following lines $(L_1): \begin{cases} x = 3+t \\ y = 5+t \\ z = 2+t \end{cases}$, and $(L_2): \begin{cases} x = -1+2s \\ y = 1+s \\ z = s \end{cases}$, $t,s \in \mathbb{R}$ are skew lines. Find the equation of the line that intersects (L_1) and (L_2) at right angles.



$\vec{P_1P_2} = (-1+2s, 1+s, s) - (3+t, 5+t, 2+t)$
$= (-4+2s-t, -4+s-t, -2+s-t)$

Solve for t and s where

$\begin{cases} \vec{P_1P_2}\cdot d_1 = 0 \\ \vec{P_1P_2}\cdot d_2 = 0 \end{cases}$

$\begin{cases} (-4+2s-t, -4+s-t, -2+s-t)\cdot(1,1,1) = 0 \\ (-4+2s-t, -4+s-t, -2+s-t)\cdot(2,1,1) = 0 \end{cases}$

$\begin{cases} 1(-4+2s-t) + 1(-4+s-t) + 1(-2+s-t) = 0 \\ 2(-4+2s-t) + 1(-4+s-t) + 1(-2+s-t) = 0 \end{cases}$

$\begin{cases} 10 = 4s-3t \\ 14 = 6s-4t \end{cases}$

$\begin{cases} 10 = 4s-3t \\ 7 = 3s-2t \end{cases}$

$\begin{bmatrix} -2 & 3 & 7 \\ -3 & 4 & 10 \end{bmatrix}$

$\sim_{-2R_2 \to R_2} \begin{bmatrix} -2 & 3 & 7 \\ 6 & -8 & -20 \end{bmatrix}$

$\sim_{3R_1+R_2 \to R_2} \begin{bmatrix} -2 & 3 & 7 \\ 0 & 1 & 1 \end{bmatrix}$

$\sim_{-3R_2+R_1 \to R_1} \begin{bmatrix} -2 & 0 & 4 \\ 0 & 1 & -1 \end{bmatrix}$

$\sim_{-\frac{1}{2}R_1 \to R_1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$

$t = -2$
$s = 1$

$\therefore\ d = \vec{P_1P_2}$ when $t=-2, s=1$

$= (-4+2(1)-(-2), -4+1-(-2), -2+1-(-2))$
$= (0, -1, 1)$

and point on $\mathcal{L}_2$:

$P = (-1+2(1), 1+(1), 1) = (1,2,1)$

$\mathcal{L}: \underset{\sim}{x} = (1,2,1) + t(0,-1,1) \quad t \in \mathbb{R}.$

Question 3. *(3 marks)* Show that the additive inverse of any vector in a vector space is unique. *Show every step, justify every step, and cite the axiom(s) used!!!*

Suppose that $\underline{w}_1$ and $\underline{w}_2$ are two distinct additive inverses of $\underline{v}$:

By axiom 5 we have $\underline{v}+\underline{w}_1=\underline{0}$
$\underline{v}+\underline{w}_2=\underline{0}$ $\Rightarrow$ $\underline{v}+\underline{w}_1=\underline{v}+\underline{w}_2$

$$\underline{w}_1+(\underline{v}+\underline{w}_1)=\underline{w}_1+(\underline{v}+\underline{w}_2)$$
$$(\underline{w}_1+\underline{v})+\underline{w}_1=(\underline{w}_1+\underline{v})+\underline{w}_2 \quad \text{by axiom 3}$$
$$\underline{0}+\underline{w}_1=\underline{0}+\underline{w}_2 \quad \text{by axiom 5}$$
$$\underline{w}_1=\underline{w}_2 \quad \text{by axiom 4}$$

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$\therefore$ additive inverses are unique.

Question 4. *(4 marks)* Let $W=\left\{A=\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in M_{22}\,\middle|\, AB=BA\right\}$ where $B=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$. Determine whether W is a subspace of M_{22}.

Lets apply the subspace test.

W is non-empty since $0\in W$ because $0B=0=B0$

1. Closure under addition

Let $A_1, A_2\in W \Rightarrow A_1B=BA_1$ and $A_2B=BA_2$

$A_1+A_2\in W$ since
$$(A_1+A_2)B=A_1B+A_2B$$
$$=BA_1+BA_2$$
$$=B(A_1+A_2)$$

2. Closure under scalar multiplication

Let $A\in W \Rightarrow AB=BA$
$r\in\mathbb{R}$

$rA\in W$ since $(rA)B=rAB=rBA=B(rA)$

$\therefore$ W is a subspace by the subspace test.

Question 5. *(2 marks)* Determine whether $U=\{(r,\,0,\,s)|rs=0 \text{ and } r,s\in\mathbb{R}\}$ is a vector space with the usual operations in $\mathbb{R}^3$.

Let $\underline{u}=(1,0,0)\in U$ since $rs=1(0)=0$
$\underline{v}=(0,0,1)\in U$ since $rs=0(1)=0$

$\underline{u}+\underline{v}=(1,0,1)\notin U$ since $rs=(1)(1)=1\neq 0$.

$\therefore$ not closed under addition

$\therefore$ U is not a vector space.