Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S5: Fall 2024: **Quiz 8** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(3 marks)* Determine whether each statement is true or false. If the statement is false provide a counterexample. If the statement is true provide a proof.

If $\mathbf{v}_1 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, then for any nonzero 2×2 matrix A, the set $\{A\mathbf{v}_1, A\mathbf{v}_2\}$ is linearly independent.

False,

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq 0$ then $A\underline{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore$ the set $\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, A\underline{v}_2 \right\}$ contains $\underline{0}$ $\therefore$ it is linearly dependent.

Question 2. *(5 marks)* Find a basis and dimension of $W = \text{span} \overbrace{\left\{ \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{M_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{M_2}, \underbrace{\begin{bmatrix} -3 & 3 \\ -6 & 0 \end{bmatrix}}_{M_3}, \underbrace{\begin{bmatrix} 8 & 7 \\ 1 & 0 \end{bmatrix}}_{M_4} \right\}}^{S}$.

Lets remove the vectors in S that make the set linearly dependent.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} -3 & 3 \\ -6 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 8 & 7 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & 8 & 0 \\ 1 & 1 & 3 & 7 & 0 \\ 1 & -1 & -6 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{matrix} -R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \end{matrix} \begin{bmatrix} 1 & 0 & -3 & 8 & 0 \\ 0 & 1 & 6 & -1 & 0 \\ 0 & -1 & -3 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim R_2 + R_3 \to R_3 \begin{bmatrix} 1 & 0 & -3 & 8 & 0 \\ 0 & 1 & 6 & -1 & 0 \\ 0 & 0 & 3 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Non-trivial solution because of the 4^{th} column. $\therefore$ if we remove the 4^{th} matrix the set becomes linearly independent and since the 4^{th} matrix can be written as a linear combination of the others we have that $S' = \{M_1, M_2, M_3\}$ spans W. and S' is a basis of W.

$\therefore \dim(W) = 3$

Question 3. *(5 marks)* Let $W = \{a+bx+cx^2+dx^3 \in P_2 \mid 2a-b+3c-d=0\}$. Find a basis and the dimension of W. Find the coordinates vector of $p(x) = 2-x+x^2+8x^3$ relative to the basis.

$b = 2a+3c-d$

$p(x) = a+bx+cx^2+dx^3$

$= a+(2a+3c-d)x+cx^2+dx^3 \in W$

$= a\underbrace{(1+2x)}_{p_1(x)} + c\underbrace{(3x+x^2)}_{p_2(x)} + d\underbrace{(x^3-x)}_{p_3(x)}$

Let $B = \{p_1(x), p_2(x), p_3(x)\}$. B spans W.

Is B linearly independent?

$\underline{0} = c_1p_1(x)+c_2p_2(x)+c_3p_3(x)$

$0+0x+0x^2 = c_1(1+2x)+c_2(3x+x^2)+c_3(x^3-x)$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}}_{A}$$

By the equivalence thm. only the trivial solution since $|A| = 1\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} = 4 \neq 0$

$\therefore$ B is a basis of W.

$\therefore$ $\dim(W) = 3$

$(2-x+x^2+8x^3)_B = (c_1, c_2, c_3)$
$= (2, 1, 8)$

$2-x+x^2+8x^3 = c_1(1+2x)+c_2(3x+x^2)$
$+c_3(x^3-x)$

Question 4. *(5 marks)* Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{p}$, $\mathbf{q}$ be nonzero vectors in $\mathbb{R}^3$, such that $\text{span}\{\mathbf{u},\mathbf{v}\} = \text{span}\{\mathbf{p},\mathbf{q}\}$. Show that $(\mathbf{u}\times\mathbf{v})\times(\mathbf{p}\times\mathbf{q}) = \mathbf{0}$.

Since $\text{span}(\{\underline{u},\underline{v}\}) = \text{span}(\{\underline{p},\underline{q}\})$ we have $\underline{u} = c_1\underline{p}+c_2\underline{q}$
$\underline{v} = k_1\underline{p}+k_2\underline{q}$

$LHS = (\underline{u}\times\underline{v})\times(\underline{p}\times\underline{q})$

$= ((c_1\underline{p}+c_2\underline{q})\times(k_1\underline{p}+k_2\underline{q}))\times(\underline{p}\times\underline{q})$

$= [(c_1\underline{p}\times k_1\underline{p})+(c_1\underline{p}\times k_2\underline{q})+(c_2\underline{q}\times k_1\underline{p})+(c_2\underline{q}\times k_2\underline{q})]\times(\underline{p}\times\underline{q})$

$= [c_1k_1\underbrace{(\underline{p}\times\underline{p})}_{\underline{0}}+c_1k_2(\underline{p}\times\underline{q})-c_2k_1(\underline{p}\times\underline{q})+c_2k_2\underbrace{(\underline{q}\times\underline{q})}_{\underline{0}}]\times(\underline{p}\times\underline{q})$

$= [(c_1k_2-c_2k_1)(\underline{p}\times\underline{q})]\times(\underline{p}\times\underline{q})$

$= (c_1k_2-c_2k_1)[(\underline{p}\times\underline{q})\times(\underline{p}\times\underline{q})]$

$= (c_1k_2-c_2k_1)\,\underline{0}$

$= \underline{0}$