Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S6: Fall 2024: **Quiz 4** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1.

a. *(3 marks)* Find all symmetric 2×2 matrices A such that $A^2 = 0$.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, for A to be symmetric $A^T = A$

$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow b = c$

$\therefore A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$

$A^2 = AA = \begin{bmatrix} a & b \\ b & d \end{bmatrix}\begin{bmatrix} a & b \\ b & d \end{bmatrix} = \begin{bmatrix} a^2+b^2 & ab+bd \\ ab+bd & b^2+d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

① $a^2+b^2=0$
② $ab+bd=0 \Rightarrow b(a+d)=0$ ②'
③ $b^2+d^2=0$

By ②' $b=0$ or $a=-d$

If $b=0 \Rightarrow a^2+0^2=0 \Rightarrow a=0$ $\therefore A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
$\Rightarrow 0^2+d^2=0 \Rightarrow d=0$

If $a=-d \Rightarrow (-d)^2+b^2=0$
$d^2+b^2=0$
$\Rightarrow d=0,\ b=0$
$\Rightarrow a=0$

$\therefore A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

b. *(2 marks)* If A is invertible and skew-symmetric ($A^T = -A$), the same is true of A^{-1}

Since A is invertible then so is A^{-1} because $AA^{-1} = I = A^{-1}A$

For A^{-1} to be skew-symmetric we need to show $(A^{-1})^T = -A^{-1}$

$LHS = (A^{-1})^T$
$= (A^T)^{-1}$
$= (-A)^{-1}$ since A is skew symmetric
$= -A^{-1}$
$= RHS$

Question 2. *(5 marks)* Given the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & -1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & -2 & 6 \\ 0 & 1 & -3 \end{bmatrix}$$

$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$

Find $\det(X)$ given that X satisfies the equation

$$(X+BC)^{-1} = A.$$

$X + BC = A^{-1}$
$X = A^{-1} - BC$

$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 4 & 0 & 0 \\ 2 & -3 & 9 \end{bmatrix}$

$X = \begin{bmatrix} -1 & 0 & 0 \\ -4 & 1 & 0 \\ -7 & 3 & -8 \end{bmatrix}$

$\det(X) = (-1)(1)(-8) = 8$

Question 3. *(3 marks)* Determine whether the following statement is true or false. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

Let A be an $n \times n$ matrix and S is an $n \times n$ invertible matrix. If $\mathbf{x}$ is a solution to the system $(S^{-1}AS)\mathbf{x} = \mathbf{b}$, then $S\mathbf{x}$ is a solution to the system $A\mathbf{y} = S\mathbf{b}$.

True, Since $\underline{x}$ is a solution of $(S^{-1}AS)\underline{x} = \underline{b}$ then it satisfies the system and we have

$$S^{-1}AS\,\underline{x} = \underline{b}$$
$$SS^{-1}AS\,\underline{x} = S\underline{b}$$
$$AS\,\underline{x} = S b \quad \clubsuit$$

$S\underline{x}$ is a solution to $A\underline{y} = S\underline{b}$ since $A\underline{y} = AS\underline{x} = S\underline{b}$ by $\clubsuit$

$\therefore$ $S\underline{x}$ satisfies the system.

Question 4. *(5 marks)* Find the real numbers x and such that

$$0 = \begin{vmatrix} 1 & x & x^2 & x^3 \\ x & x^2 & x^3 & 1 \\ x^2 & x^3 & 1 & x \\ x^3 & 1 & x & x^2 \end{vmatrix} \begin{matrix} \\ -xR_1 + R_2 \to R_2 \\ -x^2R_1 + R_3 \to R_3 \\ -x^3R_1 + R_4 \to R_4 \end{matrix} = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 0 & 0 & 1-x^4 \\ 0 & 0 & 1-x^4 & x-x^5 \\ 0 & 1-x^4 & x-x^5 & x^2-x^6 \end{vmatrix}$$

$$0 = \overset{R_2 \leftrightarrow R_4}{-} \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1-x^4 & x-x^5 & x^2-x^6 \\ 0 & 0 & 1-x^4 & x-x^5 \\ 0 & 0 & 0 & 1-x^4 \end{vmatrix}$$

$$0 = -(1-x^4)^3$$
$$0 = (1-x^4)^3$$
$$0 = 1-x^4$$
$$0 = (1-x^2)(1+x^2)$$
$$0 = (1-x)(1+x)(1+x^2)$$

$x = 1$ $\quad$ $x = -1$

Bonus. *(3 marks)* Let A and B be $m \times n$ and $n \times m$ matrices, respectively. If $m > n$, show that AB is not invertible.