Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S6: Fall 2024: **Quiz 7** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. If the volume of the parallelepiped determined by the vectors $2\mathbf{u}+3\mathbf{v}$, $3\mathbf{v}-\mathbf{w}$, and $\mathbf{u}-\mathbf{v}$ is equal to 20.

a. *(5 marks)* Find all the values of the scalar triple product $\mathbf{w}\cdot(\mathbf{v}\times\mathbf{u})$.

$$20 = \left|(\underline{u}-\underline{v})\cdot\left[(2\underline{u}+3\underline{v})\times(3\underline{v}-\underline{w})\right]\right|$$

$$\pm 20 = (\underline{u}-\underline{v})\cdot\left[(2\underline{u})\times(3\underline{v}) + (2\underline{u})\times(-\underline{w}) + \overbrace{(3\underline{v})\times(3\underline{v})}^{0} + (3\underline{v})\times(-\underline{w})\right]$$

$$\pm 20 = (\underline{u}-\underline{v})\cdot\left[6(\underline{u}\times\underline{v}) - 2(\underline{u}\times\underline{w}) - 3(\underline{v}\times\underline{w})\right]$$

$$\pm 20 = \underline{u}\cdot(6(\underline{u}\times\underline{v})) + \underline{u}\cdot(-2(\underline{u}\times\underline{w})) + \underline{u}\cdot(-3(\underline{v}\times\underline{w})) - \underline{v}\cdot(6(\underline{u}\times\underline{v})) - \underline{v}\cdot(-2\underline{u}\times\underline{w}) - \underline{v}\cdot(-3\underline{v}\times\underline{w})$$

$$\pm 20 = 6\underbrace{(\underline{u}\cdot(\underline{u}\times\underline{v}))}_{0} - 2\underbrace{\underline{u}\cdot(\underline{u}\times\underline{w})}_{0} - 3(\underline{u}\cdot(\underline{v}\times\underline{w})) - 6\,\underbrace{\underline{v}\cdot(\underline{u}\times\underline{v})}_{0} + 2\,\underline{v}\cdot(\underline{u}\times\underline{w}) + 3\,\underbrace{\underline{v}\cdot(\underline{v}\times\underline{w})}_{0}$$

$$\pm 20 = -3\,\underline{u}\cdot(\underline{v}\times\underline{w}) + 2\underline{v}\cdot(\underline{u}\times\underline{w})$$

$$\pm 20 = -3\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + 2\begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$\pm 20 = -3(-1)\underset{R_1\leftrightarrow R_3}{}\begin{vmatrix} w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} + 2(-1)\underset{R_1\leftrightarrow R_3}{}\begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\pm 20 = 3(\underline{w}\cdot(\underline{v}\times\underline{u})) + 2(-1)(-1)\underset{R_2\leftrightarrow R_3}{}\begin{vmatrix} w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

$$\pm 20 = 3(\underline{w}\cdot(\underline{v}\times\underline{u})) + 2(\underline{w}\cdot(\underline{v}\times\underline{u}))$$

$$\pm 20 = 5(\underline{w}\cdot(\underline{v}\times\underline{u}))$$

$$\pm 4 = \underline{w}\cdot(\underline{v}\times\underline{u})$$

Question 2. *(5 marks)* Show that the lines $(L_1):\begin{cases} x = 3+t \\ y = 5+t \\ z = 2+t \end{cases}$, and $(L_2):\begin{cases} x = -1+2s \\ y = 1+s \\ z = s \end{cases}$, $t,s\in\mathbb{R}$ are skew lines, and find the distance between them.

Lets show that $\mathcal{L}_1$ and $\mathcal{L}_2$ are skew lines.

$\mathcal{L}_1 \not\parallel \mathcal{L}_2$ since $\nexists k$ s.t. $\underline{d}_1 = k\underline{d}_2$.

Lets show that $\mathcal{L}_1$ and $\mathcal{L}_2$ do not intersect

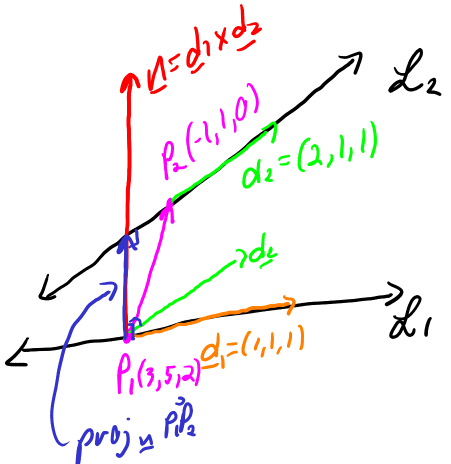
$$\left.\begin{matrix} 3+t = -1+2s \\ 5+t = 1+s \\ 2+t = s \end{matrix}\right\} \Rightarrow \begin{matrix} t-2s = -4 \\ t-s = -4 \\ t-s = -2 \end{matrix} \Rightarrow \begin{bmatrix} 1 & -2 & -4 \\ 1 & -1 & -4 \\ 1 & -1 & -2 \end{bmatrix}$$

$$\underset{\substack{-R_1+R_2\to R_2 \\ -R_1+R_3\to R_3}}{\sim}\begin{bmatrix} 1 & -2 & -4 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\underset{-R_2+R_3\to R_3}{\sim}\begin{bmatrix} 1 & -2 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The above is inconsistent. $\therefore$ no intersection between $\mathcal{L}_1$ and $\mathcal{L}_2$

$\therefore$ $\mathcal{L}_1$ and $\mathcal{L}_2$ are skew lines.



$$\underline{n} = \underline{d}_1\times\underline{d}_2 = \left(\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}\right) = (0, 1, -1)$$

$$\begin{matrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \end{matrix}$$

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = (-1,1,0) - (3,5,2) = (-4,-4,-2)$$

$$\text{proj}_{\underline{n}}\overrightarrow{P_1P_2} = \frac{\underline{n}\cdot\overrightarrow{P_1P_2}}{\underline{n}\cdot\underline{n}}\,\underline{n} = \frac{(0,1,-1)\cdot(-4,-4,-2)}{(0,1,-1)\cdot(0,1,-1)}(0,1,-1) = \frac{-4+2}{1+(-1)^2}(0,1,-1) = \frac{-2}{2}(0,1,-1) = -1(0,1,-1)$$

$\therefore$ distance $= \|\text{proj}_{\underline{n}}\overrightarrow{P_1P_2}\| = \|-1(0,1,-1)\| = \sqrt{0^2+1^2+(-1)^2} = \sqrt{2}$

Question 3. *(4 marks)* Let $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} \,\middle|\, 2a-b+3c-d=0 \right\}$. Determine whether W is a subspace of M_{22}.

W is non-empty since $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$ because $2(0)-0+3(0)-0=0$.

① Closure under addition

Let $M_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in W \Rightarrow 2a_1 - b_1 + 3c_1 - d_1 = 0$

$M_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in W \Rightarrow 2a_2 - b_2 + 3c_2 - d_2 = 0$

$M_1 + M_2 = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix} \in W$

since $2(a_1+a_2) - (b_1+b_2) + 3(c_1+c_2) - (d_1+d_2)$

$= \underbrace{2a_1 - b_1 + 3c_1 - d_1}_{0} + \underbrace{2a_2 - b_2 + 3c_2 - d_2}_{0}$

$= 0$

② Closure under scalar multiplication

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W \Rightarrow 2a - b + 3c - d = 0$

$r \in \mathbb{R}$

$rM = \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix} \in W$

since $2(ra) - (rb) + 3(rc) - (rd)$

$= r(2a - b + 3c - d)$

$= r(0)$

$= 0$

$\therefore$ W is a subspace by subspace test since closed under addition and scalar multiplication.

Question 4. *(3 marks)* Show that the additive inverse of any vector in a vector space is unique. *Show every step, justify every step, and cite the axiom(s) used!!!*

Suppose that $\underline{w}_1$ and $\underline{w}_2$ are two distinct additive inverses of $\underline{v}$:

By axiom ⑤ we have $\underline{v} + \underline{w}_1 = \underline{0}$, $\underline{v} + \underline{w}_2 = \underline{0}$ $\Rightarrow$ $\underline{v} + \underline{w}_1 = \underline{v} + \underline{w}_2$

$\underline{w}_1 + (\underline{v} + \underline{w}_1) = \underline{w}_1 + (\underline{v} + \underline{w}_2)$

$(\underline{w}_1 + \underline{v}) + \underline{w}_1 = (\underline{w}_1 + \underline{v}) + \underline{w}_2$ by axiom ③

$\underline{0} + \underline{w}_1 = \underline{0} + \underline{w}_2$ by axiom ⑤

$\underline{w}_1 = \underline{w}_2$ by axiom ④

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$\therefore$ additive inverses are unique.

Question 5. *(2 marks)* Determine whether $U = \{(r,\ 0,\ s) | r^2 + s^2 = 2 \text{ and } r,\ s \in \mathbb{R}\}$ is a vector space with the usual operations in $\mathbb{R}^3$.

U is not a vector space since $\underline{u} = (1,0,1) \in U$ since $1^2 + 1^2 = 2$

but $\underline{u} + \underline{u} = (2,0,2) \notin U$ since $2^2 + 2^2 \neq 2$

$\therefore$ not closed under addition.