Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S6: Fall 2024: **Quiz 8** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(3 marks)* Determine whether each statement is true or false. If the statement is false provide a counterexample. If the statement is true provide a proof.

If $\mathbf{v}_1 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, then for any nonzero 2×2 matrix A, the set $\{A\mathbf{v}_1, A\mathbf{v}_2\}$ is linearly independent.

False,

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq 0$ then $A\underline{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore$ the set $\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, A\underline{v}_2 \right\}$ contains $\underline{0}$ $\therefore$ it is linearly dependent.

Question 2. *(5 marks)* Find a basis and dimension of $W = \text{span}\,\overbrace{\{\underbrace{x^2 - x + 2}_{p_1(x)},\ \underbrace{x - 1}_{p_2(x)},\ \underbrace{-3x^2 + 3x - 6}_{p_3(x)},\ \underbrace{8x^2 + 7x + 1}_{p_4(x)}\}}^{S}$.

Lets remove the vectors in that make the set linearly dependent. $\underline{0} = c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x) + c_4 p_4(x)$

$$\begin{bmatrix} 1 & 0 & -3 & 8 & 0 \\ -1 & 1 & 3 & 7 & 0 \\ 2 & -1 & -6 & 1 & 0 \end{bmatrix} \sim \begin{matrix} R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{matrix} \begin{bmatrix} 1 & 0 & -3 & 8 & 0 \\ 0 & 1 & 0 & 15 & 0 \\ 0 & -1 & 0 & -15 & 0 \end{bmatrix} \sim_{R_2 + R_3 \to R_3} \begin{bmatrix} 1 & 0 & -3 & 8 & 0 \\ 0 & 1 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Non-trivial solution because of the 3^{rd} & 4^{th} column. $\therefore$ if we remove the 3^{rd} & 4^{th} pol. the set becomes linearly independent and since the 3^{rd} & 4^{th} pol. can be written as a linear combination of the others, we have that $S' = \{p_1(x), p_2(x)\}$ spans W. and S' is a basis of W.

$\therefore \dim(W) = 2$

Question 3. *(5 marks)*
Let $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} \,\middle|\, 2a - b + 3c - d = 0 \right\}$. Find a basis and the dimension of W. Find the coordinates vector of $A = \begin{bmatrix} 2 & -1 \\ 1 & 8 \end{bmatrix}$ relative to the basis.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= \begin{bmatrix} a & 2a+3c-d \\ c & d \end{bmatrix} \in W$

$= a \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}}_{M_1} + c \underbrace{\begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}}_{M_2} + d \underbrace{\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}}_{M_3}$

Let $\beta = \{M_1, M_2, M_3\}$. $\therefore$ β spans W

Is β linearly independent?

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow c_1 = 0$
$c_2 = 0$
$c_3 = 0$

$\therefore$ only trivial linear combination.

$\therefore$ β is linearly independent $\therefore$ β is a basis.

$\left(\begin{bmatrix} 2 & -1 \\ 1 & 8 \end{bmatrix}\right)_\beta = (c_1, c_2, c_3)$
$= (2, 1, 8)$

$\begin{bmatrix} 2 & -1 \\ 1 & 8 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$

Question 4. *(5 marks)* Suppose that $U = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ where each $\mathbf{x}_i$ is in $\mathbb{R}^n$. If A is an $m \times n$ matrix and $A\mathbf{x}_i = \mathbf{0}$ for each i, show that $A\mathbf{y} = \mathbf{0}$ for every vector $\mathbf{y}$ in U.

Let $\underline{y} \in \text{span}(\{\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_k\}) \Rightarrow \underline{y} = c_1\underline{x}_1 + c_2\underline{x}_2 + \cdots + c_k\underline{x}_k$

$Ay = A(c_1\underline{x}_1 + c_2\underline{x}_2 + \cdots + c_k\underline{x}_k)$
$= Ac_1\underline{x}_1 + Ac_2\underline{x}_2 + \cdots + Ac_k\underline{x}_k$
$= c_1A\underline{x}_1 + c_2A\underline{x}_2 + \cdots + c_kA\underline{x}_k$
$= c_1\underline{0} + c_2\underline{0} + \cdots + c_k\underline{0}$ by the premise $A\underline{x}_i = \underline{0}$
$= \underline{0}$