Show all your work and justify your answers.

Note: This quiz is representative of the difficulty level of Test 1, but covers only a small portion of the content that will be assessed.

Question 1. (5 marks) Solve for the matrix A where

$$A^T \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - 2(A^{-1})^T \right)^{-1}.$$

Let $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$A^TB = \left(C - 2(A^{-1})^T\right)^{-1}$

$(A^TB)^{-1} = \left[\left(C - 2(A^{-1})^T\right)^{-1}\right]^{-1}$

$B^{-1}(A^T)^{-1} = C - 2(A^{-1})^T$

$B^{-1}(A^T)^{-1} + 2(A^{-1})^T = C$

$B^{-1}(A^T)^{-1} + 2(A^T)^{-1} = C$

$(B^{-1} + 2I)(A^T)^{-1} = C$

$(B^{-1} + 2I)(A^T)^{-1}(A^T) = C(A^T)$

$B^{-1} + 2I = CA^T$

$C^{-1}(B^{-1} + 2I) = C^{-1}CA^T$

$C^{-1}(B^{-1} + 2I) = IA^T$

$C^{-1}(B^{-1} + 2I) = A^T$

$(C^{-1}(B^{-1} + 2I))^T = A$

$A = \left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right)^T$

$A = \left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -2 & 3 \end{bmatrix} \right)^T$

$A = \begin{bmatrix} 3 & 0 \\ -5 & 3 \end{bmatrix}^T$

$A = \begin{bmatrix} 3 & -5 \\ 0 & 3 \end{bmatrix}$

Question 2. (5 marks) Show that every matrix A can be factored as $A = UR$, where U is invertible and R is in reduced row-echelon form.

Perform Gauss-Jordan on A

$A \sim$ k elementary row operations $\sim R$ where R is the RREF of A.

For each of the k elementary row operations we can obtain an elementary matrix E_i s.t.

$$R = E_k \cdots E_2 E_1 A$$

since elementary matrices are invertible then so is their product

$(E_k \cdots E_2 E_1)^{-1} R = (E_k \cdots E_2 E_1)^{-1} E_k \cdots E_2 E_1 A$

$(E_k \cdots E_2 E_1)^{-1} R = IA$

$U R = A$

where $U = (E_k \cdots E_2 E_1)^{-1}$ an invertible matrix.

Question 3. (5 marks) Find a sequence of elementary row operations that brings

$$\begin{bmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ c_1+a_1 & c_2+a_2 & c_3+a_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

$\sim -R_1+R_2 \to R_2 \begin{bmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1-b_1 & a_2-b_2 & a_3-b_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{bmatrix}$

$\sim R_3+R_2 \to R_2 \begin{bmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ 2a_1 & 2a_2 & 2a_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{bmatrix}$

$\sim \frac{1}{2} R_2 \to R_2 \begin{bmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1 & a_2 & a_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{bmatrix}$

$\sim -R_2+R_3 \to R_3 \begin{bmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$

$\sim -R_3+R_1 \to R_1 \begin{bmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$

$\sim R_1 \leftrightarrow R_2 \begin{bmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \sim R_2 \leftrightarrow R_3 \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$