Dawson College: Linear Algebra and Vector Geometry
201-SN4-RE-S03: Fall 2026
Test 1: Worked Solutions

Question 1. *(1 mark each)* Complete each sentence with **MUST**, **MIGHT**, or **CANNOT**.

a. Let A be a square matrix. If $A\mathbf{x} = A\mathbf{y}$ only when $\mathbf{x} = \mathbf{y}$, then A _____ be invertible.

Solution.
MUST.

b. A linear system with three equations and two variables _____ be inconsistent.

Solution.
MIGHT.

c. If A is a product of elementary matrices, then A _____ be singular.

Solution.
CANNOT.

d. If A and B are invertible $n \times n$ matrices, then $A + B$ _____ be invertible.

Solution.
MIGHT.

e. If A, B, and C are square matrices of the same size and $ABC^2 = I$, then C _____ be invertible.

Solution.
MUST.

Question 2. *(3 marks each)* Determine whether each statement is true or false. If a statement is false, provide a counterexample. If it is true, provide a proof.

a. Consider a system of linear equations with augmented matrix A. If there is more than one solution, then A has a row of zeros.

Solution.
False. Consider

$$\begin{cases} x + y = 1, \\ 2x + 2y = 2. \end{cases} \qquad A = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}\right].$$

The augmented matrix has no row of zeros. Nevertheless,

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}\right] \overset{-2R_1+R_2\to R_2}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right].$$

Letting $y = t$ gives $(x, y) = (1 - t, t)$, $t \in \mathbb{R}$, so there are infinitely many solutions. A zero row in an echelon form need not already occur in the original augmented matrix.

b. If A^2 is a symmetric matrix, then A is a symmetric matrix.

Solution.

False. Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \qquad A^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq A.$$

However,

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which is symmetric. Therefore A^2 can be symmetric when A is not.

c. If A and B are square matrices of the same size, then $A^TB - B^TA$ is skew-symmetric.

Solution.

True. Let $M = A^TB - B^TA$. To show that M is skew-symmetric, we must show that $M^T = -M$. Using the transpose rules,

$$\begin{aligned} M^T &= (A^TB - B^TA)^T \\ &= (A^TB)^T - (B^TA)^T \\ &= B^T(A^T)^T - A^T(B^T)^T \\ &= B^TA - A^TB \\ &= -(A^TB - B^TA) \\ &= -M. \end{aligned}$$

Therefore $A^TB - B^TA$ is skew-symmetric.

Question 3. Given that X, A, B, and C are $n \times n$ matrices, solve the following equation for X. Assume any necessary matrices to be invertible. *(5 marks)*

$$2X^T + A = (XB - C)^T$$

Solution.

Transpose both sides, using $(UV)^T = V^TU^T$ and $(X^T)^T = X$:

$$\begin{aligned} 2X^T + A &= (XB - C)^T, \\ 2X + A^T &= XB - C, \\ A^T + C &= XB - 2X, \\ A^T + C &= X(B - 2I). \end{aligned}$$

Multiply on the *right* by $(B - 2I)^{-1}$:

$$\begin{aligned} (A^T + C)(B - 2I)^{-1} &= X(B - 2I)(B - 2I)^{-1} \\ &= XI = X. \end{aligned}$$

Hence

$$\boxed{X = (A^T + C)(B - 2I)^{-1}}.$$

Question 4. *(6 marks)*

Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & k & 6 \\ 0 & 2 & 4k \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 3h \\ 2 \end{bmatrix}.$$

Find the values of h and k, if possible, for which the equation $A\mathbf{x} = \mathbf{b}$ has:

a. a unique solution;
b. infinitely many solutions;
c. no solution.

Solution.

Row-reduce without dividing by a parameter:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ -1 & k & 6 & 3h \\ 0 & 2 & 4k & 2 \end{array}\right] \overset{R_1+R_2\to R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & k & 8 & 3h+1 \\ 0 & 2 & 4k & 2 \end{array}\right]$$

$$\overset{R_2\leftrightarrow R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 2 & 4k & 2 \\ 0 & k & 8 & 3h+1 \end{array}\right]$$

$$\overset{\substack{2R_3\to R_3 \\ -kR_2+R_3\to R_3}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 2 & 4k & 2 \\ 0 & 0 & 16-4k^2 & 6h+2-2k \end{array}\right].$$

The first two leading entries are nonzero for every k. The last coefficient is $4(4-k^2)$ and vanishes exactly when $k = 2$ or $k = -2$.

a. **Unique solution:** if $k \neq \pm 2$, there is a leading entry in every variable column and no inconsistent row. Therefore $\boxed{k \neq \pm 2,\ h \in \mathbb{R}}$.

b. **Infinitely many solutions:** when $k = 2$, the last row is $[\,0\ 0\ 0 \mid 6h-2\,]$, so consistency requires $h = \frac{1}{3}$. When $k = -2$, it is $[\,0\ 0\ 0 \mid 6h+6\,]$, so consistency requires $h = -1$. In each consistent case there are two leading entries and one free variable. Hence

$$\boxed{(k,h) = (2, \frac{1}{3}) \text{ or } (-2,-1)}.$$

c. **No solution:** an exceptional value of k with a nonzero last entry makes the system inconsistent. Therefore

$$\boxed{k = 2,\ h \neq \frac{1}{3} \quad \text{or} \quad k = -2,\ h \neq -1}.$$

Question 5.

a. Use the inversion algorithm to find B^{-1}, where *(5 marks)*

$$B = \begin{bmatrix} 2 & 3 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

b. Using the inverse found in part (a), solve the linear system

$$\begin{cases} 2x + 3y = -1, \\ 4x + 5y = -1, \\ z = 2. \end{cases}$$

(2 marks)

Solution.

a. Apply the inversion algorithm to $[B \mid I]$:

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 4 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \overset{-2R_1+R_2\to R_2}{\sim} \left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{-R_2\to R_2}{\sim} \left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{-3R_2+R_1\to R_1}{\sim} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -5 & 3 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{\frac{1}{2}R_1\to R_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{2} & \frac{3}{2} & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

The left block is I_3, so

$$\boxed{B^{-1} = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$

b. Write the system as $B\mathbf{x} = \mathbf{b}$. Then

$$\mathbf{x} = B^{-1}\mathbf{b} = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} - \frac{3}{2} \\ -2 + 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Therefore $\boxed{x = 1,\ y = -1,\ z = 2}$.

Question 6. Let

$$A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}, \qquad B = \begin{bmatrix} 2d & b + 2d & -3c & a + 3c \\ 2h & f + 2h & -3g & e + 3g \\ 2l & j + 2l & -3k & i + 3k \\ 2p & n + 2p & -3o & m + 3o \end{bmatrix}.$$

Given that $\det(A) = 5$, compute $\det(B)$. Show your work and explain how the changes from A to B affect the determinant. *(5 marks)*

Solution.

Use elementary column operations to recover A, carrying the determinant factors outside.

$$\det(B) = \begin{vmatrix} 2d & b+2d & -3c & a+3c \\ 2h & f+2h & -3g & e+3g \\ 2l & j+2l & -3k & i+3k \\ 2p & n+2p & -3o & m+3o \end{vmatrix}$$

$$\overset{-C_1+C_2\to C_2}{=} \begin{vmatrix} 2d & b & -3c & a+3c \\ 2h & f & -3g & e+3g \\ 2l & j & -3k & i+3k \\ 2p & n & -3o & m+3o \end{vmatrix}$$

$$\overset{C_3+C_4\to C_4}{=} \begin{vmatrix} 2d & b & -3c & a \\ 2h & f & -3g & e \\ 2l & j & -3k & i \\ 2p & n & -3o & m \end{vmatrix}$$

$$\overset{\substack{\frac{1}{2}C_1\to C_1 \\ -\frac{1}{3}C_3\to C_3}}{=} -6\begin{vmatrix} d & b & c & a \\ h & f & g & e \\ l & j & k & i \\ p & n & o & m \end{vmatrix}$$

$$\overset{C_1\leftrightarrow C_4}{=} 6\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix}$$

$$= 6\det(A) = 6(5) = \boxed{30}.$$

Adding a multiple of another column does not change the determinant. The two scalings contribute a factor of -6 outside, and the interchange changes its sign.

Question 7. Compute the following determinant. Show all your work. *(5 marks)*

$$\begin{vmatrix} 1 & 2 & 2 & 0 \\ 2 & 1 & -1 & 0 \\ 3 & 0 & 1 & 2 \\ 1 & 3 & 2 & -2 \end{vmatrix}.$$

Solution.

Expand along the fourth column, which contains two zeros:

$$D = 2(-1)^{3+4}\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & -1 \\ 1 & 3 & 2 \end{vmatrix} + (-2)(-1)^{4+4}\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & -1 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & -1 \\ 1 & 3 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & -1 \\ 3 & 0 & 1 \end{vmatrix}.$$

Expand each minor along its first row:

$$\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & -1 \\ 1 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}$$
$$= (2+3) - 2(4+1) + 2(6-1) = 5,$$

$$\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & -1 \\ 3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix}$$
$$= (1-0) - 2(2+3) + 2(0-3) = -15.$$

Therefore

$$\boxed{D = -2(5) - 2(-15) = 20}.$$

Question 8. Find all matrices A, if any, such that *(5 marks)*

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} A = I_2.$$

Solution.

The matrix A must be 3×2. Write

$$A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}.$$

Multiplying and equating corresponding entries gives

$$\begin{bmatrix} a+c+e & b+d+f \\ c+e & d+f \end{bmatrix} = I_2, \qquad \begin{cases} a+c+e = 1, \\ b+d+f = 0, \\ c+e = 0, \\ d+f = 1. \end{cases}$$

Use the variable order a, b, c, d, e, f. Gauss–Jordan elimination gives

$$\left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \overset{\substack{-R_3+R_1\to R_1 \\ -R_4+R_2\to R_2}}{\sim} \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right].$$

The matrix is in reduced row echelon form. Let $e = s$ and $f = t$. Then $a = 1$, $b = -1$, $c = -s$, and $d = 1 - t$. Therefore

$$\boxed{A = \begin{bmatrix} 1 & -1 \\ -s & 1-t \\ s & t \end{bmatrix}, \qquad s, t \in \mathbb{R}}.$$

Question 9. Find two elementary matrices E_1 and E_2 that satisfy *(4 marks)*

$$E_2E_1\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 5\\ -2 & -2 & -11\end{bmatrix} = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 5\\ 0 & 0 & 1\end{bmatrix}.$$

Solution.

The first two rows are to remain unchanged. Eliminate the first and second entries of the third row:

$$\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 5\\ -2 & -2 & -11\end{bmatrix} \overset{2R_1+R_3\to R_3}{\sim} \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 5\\ 0 & -2 & -9\end{bmatrix} \overset{2R_2+R_3\to R_3}{\sim} \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 5\\ 0 & 0 & 1\end{bmatrix}.$$

Apply each operation separately to I_3 to obtain its elementary matrix:

$$\boxed{E_1 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 2 & 0 & 1\end{bmatrix}, \qquad E_2 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 2 & 1\end{bmatrix}.}$$

Question 10. Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system of n linear equations in n unknowns, and let Q be an invertible $n \times n$ matrix. Using the Equivalence Theorem, prove that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if $(QA)\mathbf{x} = \mathbf{0}$ has only the trivial solution. *(5 marks)*

Solution.

By the Equivalence Theorem, a square matrix is invertible if and only if its homogeneous system has only the trivial solution.

Forward direction. Suppose $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Then A is invertible. Since Q is invertible, QA is invertible. Applying the Equivalence Theorem to QA, the system $(QA)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Reverse direction. Suppose $(QA)\mathbf{x} = \mathbf{0}$ has only the trivial solution. Then QA is invertible. Since

$$A = Q^{-1}(QA),$$

A is a product of invertible matrices and is invertible. The Equivalence Theorem now implies that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Both directions have been proved. $\square$