Dawson College: Linear Algebra and Vector Geometry
201-SN4-RE-S04, Version A: Fall 2026
Test 1: Worked Solutions

Question 1. *(1 mark each)* Complete each sentence with **MUST**, **MIGHT**, or **CANNOT**.

a. If A is upper triangular, then A^T _____ be upper triangular.

Solution.
MIGHT.

b. A linear system with fewer equations than variables _____ have exactly one solution.

Solution.
CANNOT.

c. If E_1 and E_2 are elementary matrices of the same size, then E_1E_2 _____ be an elementary matrix.

Solution.
MIGHT.

d. If A is invertible, then A^2 _____ be invertible.

Solution.
MUST.

e. If A and B are symmetric matrices of the same size, then AB _____ be symmetric.

Solution.
MIGHT.

Question 2. *(3 marks each)* Determine whether each statement is true or false. If a statement is false, provide a counterexample. If it is true, provide a proof.

a. If two square matrices have the same reduced row echelon form, then they have the same determinant.

Solution.
False. Take

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Both have reduced row echelon form I_2:

$$B \overset{\frac{1}{2}R_1 \to R_1}{\sim} I_2.$$

But $\det(A) = 1$ and $\det(B) = 2$. Row-equivalent matrices need not have the same determinant.

b. If A is a square matrix and $A^2 = I$, then $A = I$ or $A = -I$.

Solution.
False. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2,$$

but $A \neq I_2$ and $A \neq -I_2$. A matrix whose square is I need not be one of the two scalar matrices I and $-I$.

c. If A is skew-symmetric and B is a square matrix of the same size, then B^TAB is skew-symmetric.

Solution.

True. Since A is skew-symmetric, $A^T = -A$. Let $M = B^TAB$. Then

$$\begin{aligned} M^T &= (B^TAB)^T \\ &= B^TA^T(B^T)^T \\ &= B^T(-A)B \\ &= -B^TAB = -M. \end{aligned}$$

Therefore B^TAB is skew-symmetric.

Question 3. Given that X, A, B, and C are $n \times n$ matrices, solve the following equation for X. Assume any necessary matrices to be invertible. *(5 marks)*

$$3X^T + B = (CX - A)^T$$

Solution.

Transpose both sides and use $(X^T)^T = X$:

$$\begin{aligned} 3X^T + B &= (CX - A)^T, \\ 3X + B^T &= CX - A, \\ A + B^T &= CX - 3X, \\ A + B^T &= (C - 3I)X. \end{aligned}$$

Multiply on the left by $(C - 3I)^{-1}$:

$$\boxed{X = (C - 3I)^{-1}(A + B^T)}.$$

Question 4. *(6 marks)*

Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & k & 3 \\ 0 & 2 & 2k \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2h \\ 2 \end{bmatrix}.$$

Find the values of h and k, if possible, for which the equation $A\mathbf{x} = \mathbf{b}$ has:

a. a unique solution;
b. infinitely many solutions;
c. no solution.

Solution.

Row-reduce without dividing by a parameter:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ -1 & k & 3 & 2h \\ 0 & 2 & 2k & 2 \end{array}\right] \overset{R_1+R_2\to R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & k & 4 & 2h+1 \\ 0 & 2 & 2k & 2 \end{array}\right]$$

$$\overset{R_2\leftrightarrow R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 2 & 2k & 2 \\ 0 & k & 4 & 2h+1 \end{array}\right]$$

$$\overset{\substack{2R_3\to R_3 \\ -kR_2+R_3\to R_3}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 2 & 2k & 2 \\ 0 & 0 & 8-2k^2 & 4h+2-2k \end{array}\right].$$

The first two leading entries are nonzero for every k. The last coefficient is $2(4-k^2)$ and vanishes exactly when $k=2$ or $k=-2$.

a. **Unique solution:** if $k \neq \pm 2$, there is a leading entry in every variable column and no inconsistent row. Therefore $\boxed{k \neq \pm 2,\ h \in \mathbb{R}}$.

b. **Infinitely many solutions:** when $k=2$, the last row is $[\,0\ 0\ 0 \mid 4h-2\,]$, so consistency requires $h=\frac{1}{2}$. When $k=-2$, it is $[\,0\ 0\ 0 \mid 4h+6\,]$, so consistency requires $h=-\frac{3}{2}$. In each consistent case there are two leading entries and one free variable. Hence

$$\boxed{(k,h)=(2,\frac{1}{2}) \text{ or } (-2,-\frac{3}{2})}.$$

c. **No solution:** an exceptional value of k with a nonzero last entry makes the system inconsistent. Therefore

$$\boxed{k=2,\ h\neq\frac{1}{2} \quad \text{or} \quad k=-2,\ h\neq-\frac{3}{2}}.$$

Question 5.

a. Use the inversion algorithm to find B^{-1}, where *(5 marks)*

$$B=\begin{bmatrix}3&1&0\\2&1&0\\0&0&2\end{bmatrix}.$$

b. Using the inverse found in part (a), solve the linear system

$$\begin{cases}3x+y=2,\\2x+y=1,\\2z=4.\end{cases}$$

(2 marks)

Solution.

a. Apply the inversion algorithm to $[B \mid I]$:

$$\left[\begin{array}{ccc|ccc}3&1&0&1&0&0\\2&1&0&0&1&0\\0&0&2&0&0&1\end{array}\right] \underset{-R_2+R_1\to R_1}{\sim} \left[\begin{array}{ccc|ccc}1&0&0&1&-1&0\\2&1&0&0&1&0\\0&0&2&0&0&1\end{array}\right]$$

$$\underset{-2R_1+R_2\to R_2}{\sim} \left[\begin{array}{ccc|ccc}1&0&0&1&-1&0\\0&1&0&-2&3&0\\0&0&2&0&0&1\end{array}\right]$$

$$\underset{\frac{1}{2}R_3\to R_3}{\sim} \left[\begin{array}{ccc|ccc}1&0&0&1&-1&0\\0&1&0&-2&3&0\\0&0&1&0&0&\frac{1}{2}\end{array}\right].$$

The left block is I_3, so

$$\boxed{B^{-1}=\begin{bmatrix}1&-1&0\\-2&3&0\\0&0&\frac{1}{2}\end{bmatrix}}.$$

b. Since $B\mathbf{x} = \mathbf{b}$,

$$\mathbf{x} = B^{-1}\mathbf{b} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2-1 \\ -4+3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Therefore $\boxed{x = 1,\ y = -1,\ z = 2}$.

Question 6. Let

$$A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}, \qquad B = \begin{bmatrix} -2c & 3b & a-2c & d-3b \\ -2g & 3f & e-2g & h-3f \\ -2k & 3j & i-2k & l-3j \\ -2o & 3n & m-2o & p-3n \end{bmatrix}.$$

Given that $\det(A) = -4$, compute $\det(B)$. Show your work and explain how the changes from A to B affect the determinant. *(5 marks)*

Solution.

Use elementary column operations to recover A, carrying the determinant factors outside.

$$\begin{aligned} \det(B) &= \begin{vmatrix} -2c & 3b & a-2c & d-3b \\ -2g & 3f & e-2g & h-3f \\ -2k & 3j & i-2k & l-3j \\ -2o & 3n & m-2o & p-3n \end{vmatrix} \\ &\overset{-C_1+C_3\to C_3}{=} \begin{vmatrix} -2c & 3b & a & d-3b \\ -2g & 3f & e & h-3f \\ -2k & 3j & i & l-3j \\ -2o & 3n & m & p-3n \end{vmatrix} \\ &\overset{C_2+C_4\to C_4}{=} \begin{vmatrix} -2c & 3b & a & d \\ -2g & 3f & e & h \\ -2k & 3j & i & l \\ -2o & 3n & m & p \end{vmatrix} \\ &\overset{\substack{-\frac{1}{2}C_1\to C_1 \\ \frac{1}{3}C_2\to C_2}}{=} -6\begin{vmatrix} c & b & a & d \\ g & f & e & h \\ k & j & i & l \\ o & n & m & p \end{vmatrix} \\ &\overset{C_1\leftrightarrow C_3}{=} 6\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} \\ &= 6\det(A) = 6(-4) = \boxed{-24}. \end{aligned}$$

Adding a multiple of another column does not change the determinant. The two scalings contribute a factor of -6 outside, and the interchange changes its sign.

Question 7. Compute the following determinant. Show all your work. *(5 marks)*

$$\begin{vmatrix} 2 & 0 & 1 & 3 \\ 1 & 0 & 2 & -1 \\ 3 & 2 & 0 & 1 \\ 1 & -2 & 3 & 2 \end{vmatrix}.$$

Solution.

Expand along column 2, which contains two zeros:

$$D = -2\begin{vmatrix} 2 & 1 & 3 \\ 1 & 2 & -1 \\ 1 & 3 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 & 3 \\ 1 & 2 & -1 \\ 3 & 0 & 1 \end{vmatrix}.$$

Expand each minor along its first row:

$$\begin{aligned} \begin{vmatrix} 2 & 1 & 3 \\ 1 & 2 & -1 \\ 1 & 3 & 2 \end{vmatrix} &= 2\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \\ &= 14. \end{aligned}$$

$$\begin{aligned} \begin{vmatrix} 2 & 1 & 3 \\ 1 & 2 & -1 \\ 3 & 0 & 1 \end{vmatrix} &= 2\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \\ &= -18. \end{aligned}$$

Therefore

$$\boxed{D = -2(14) - 2(-18) = 8}.$$

Question 8. Find all matrices X, if any, such that *(5 marks)*

$$X\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} = I_2.$$

Solution.

The matrix X must be 2×3. Write

$$X = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}.$$

Multiplying and equating corresponding entries gives

$$\begin{bmatrix} a+2b+c & b+c \\ d+2e+f & e+f \end{bmatrix} = I_2, \qquad \begin{cases} a+2b+c=1, \\ b+c=0, \\ d+2e+f=0, \\ e+f=1. \end{cases}$$

Use the variable order a, b, c, d, e, f. Gauss–Jordan elimination gives

$$\left[\begin{array}{cccccc|c} 1 & 2 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right] \overset{\substack{-2R_2+R_1\to R_1 \\ -2R_4+R_3\to R_3}}{\sim} \left[\begin{array}{cccccc|c} 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right].$$

The matrix is in reduced row echelon form. Let $c = s$ and $f = t$. Then $a = 1 + s$, $b = -s$, $d = t - 2$, and $e = 1 - t$. Therefore

$$\boxed{X = \begin{bmatrix} 1+s & -s & s \\ t-2 & 1-t & t \end{bmatrix}, \qquad s, t \in \mathbb{R}}.$$

Question 9. Find two elementary matrices E_1 and E_2 that satisfy the following equation. *(4 marks)*

$$E_2 E_1 \begin{bmatrix} -2 & -3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}.$$

Solution.

Eliminate the first two entries of R_1 using R_2 and R_3:

$$\begin{bmatrix} -2 & -3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \underset{\sim}{2R_2+R_1 \to R_1} \begin{bmatrix} 0 & -3 & 4 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\underset{\sim}{3R_3+R_1 \to R_1} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}.$$

Apply each operation separately to I_3 to obtain

$$\boxed{E_1 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$

Question 10. Let A and B be $m \times n$ matrices. Prove that A and B are row equivalent if and only if they have the same reduced row echelon form. *(5 marks)*

Solution.

Forward direction. Suppose A and B are row equivalent. Let R be the reduced row echelon form of A. A sequence of elementary row operations transforms B into A, and a second sequence transforms A into R:

$$B \sim A \sim R.$$

Therefore R is a reduced row echelon form of B as well. By uniqueness of reduced row echelon form, A and B have the same reduced row echelon form.

Reverse direction. Suppose A and B have the same reduced row echelon form R. There are sequences of elementary row operations

$$A \sim R \qquad \text{and} \qquad B \sim R.$$

Every elementary row operation can be reversed using its inverse elementary row operation. Reverse the sequence from B to R to obtain a sequence from R to B. Combining sequences gives

$$A \sim R \sim B.$$

Therefore A and B are row equivalent. Both directions are proved. □