Dawson College: Linear Algebra and Vector Geometry
201-SN4-RE-S04, Version B: Fall 2026
Test 1: Worked Solutions

Question 1. *(1 mark each)* Complete each sentence with **MUST**, **MIGHT**, or **CANNOT**.

a. If a linear system has two distinct solutions, then it _____ have infinitely many solutions.

Solution.
MUST.

b. A linear system whose augmented matrix has a leading entry in its last column _____ be consistent.

Solution.
CANNOT.

c. If $A + B$ is invertible, then A _____ be invertible.

Solution.
MIGHT.

d. If A is a square matrix and A^2 is invertible, then A _____ be invertible.

Solution.
MUST.

e. A square matrix all of whose entries are nonzero _____ have determinant zero.

Solution.
MIGHT.

Question 2. *(3 marks each)* Determine whether each statement is true or false. If a statement is false, provide a counterexample. If it is true, provide a proof.

a. If A is an $n \times n$ matrix and $A\mathbf{x} = \mathbf{b}$ is inconsistent for one vector $\mathbf{b}$, then it is inconsistent for every nonzero vector $\mathbf{b}$.

Solution.
False. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

For $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the second equation is $0 = 1$, so the system is inconsistent. However, for the nonzero vector $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the system is

$$x = 1, \qquad 0 = 0,$$

and has solutions $(x, y) = (1, t)$, $t \in \mathbb{R}$.

b. If A is a square matrix and $A^2 = A$, then $A = 0$ or $A = I$.

Solution.

False. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A,$$

although $A \neq 0$ and $A \neq I_2$.

c. If A and B are symmetric matrices of the same size, then $AB - BA$ is skew-symmetric.

Solution.

True. Since A and B are symmetric, $A^T = A$ and $B^T = B$. Let $M = AB - BA$. Then

$$\begin{aligned} M^T &= (AB - BA)^T \\ &= B^T A^T - A^T B^T \\ &= BA - AB \\ &= -(AB - BA) = -M. \end{aligned}$$

Therefore $AB - BA$ is skew-symmetric.

Question 3. Given that X, A, B, and C are $n \times n$ matrices, solve the following equation for X. Assume any necessary matrices to be invertible. *(5 marks)*

$$(XB)^T - 2X^T = CA^T$$

Solution.

Transpose both sides and use $(UV)^T = V^T U^T$:

$$\begin{aligned} (XB)^T - 2X^T &= CA^T, \\ XB - 2X &= AC^T, \\ X(B - 2I) &= AC^T. \end{aligned}$$

Multiply on the right by $(B - 2I)^{-1}$:

$$\boxed{X = AC^T(B - 2I)^{-1}}.$$

Question 4. *(6 marks)*

Let

$$A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & k & 1 \\ 0 & 2 & 2k \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 4h \\ 6 \end{bmatrix}.$$

Find the values of h and k, if possible, for which the equation $A\mathbf{x} = \mathbf{b}$ has:

a. a unique solution;
b. infinitely many solutions;
c. no solution.

Solution.

Row-reduce without dividing by a parameter:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ -1 & k & 1 & 4h \\ 0 & 2 & 2k & 6 \end{array}\right] \overset{R_1+R_2\to R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & k & 4 & 4h+2 \\ 0 & 2 & 2k & 6 \end{array}\right]$$

$$\overset{R_2\leftrightarrow R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 2 & 2k & 6 \\ 0 & k & 4 & 4h+2 \end{array}\right]$$

$$\overset{\substack{2R_3\to R_3 \\ -kR_2+R_3\to R_3}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 2 & 2k & 6 \\ 0 & 0 & 8-2k^2 & 8h+4-6k \end{array}\right].$$

The first two leading entries are nonzero for every k. The last coefficient is $2(4-k^2)$ and vanishes exactly when $k=2$ or $k=-2$.

a. **Unique solution:** if $k \neq \pm 2$, there is a leading entry in every variable column and no inconsistent row. Therefore $\boxed{k \neq \pm 2,\ h \in \mathbb{R}}$.

b. **Infinitely many solutions:** when $k=2$, the last row is $[\,0\ 0\ 0 \mid 8h-8\,]$, so consistency requires $h=1$. When $k=-2$, it is $[\,0\ 0\ 0 \mid 8h+16\,]$, so consistency requires $h=-2$. In each consistent case there are two leading entries and one free variable. Hence

$$\boxed{(k,h) = (2,1) \text{ or } (-2,-2)}.$$

c. **No solution:** an exceptional value of k with a nonzero last entry makes the system inconsistent. Therefore

$$\boxed{k=2,\ h\neq 1 \quad \text{or} \quad k=-2,\ h\neq -2}.$$

Question 5.

a. Use the inversion algorithm to find B^{-1}, where *(5 marks)*

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

b. Using the inverse found in part (a), solve the linear system

$$\begin{cases} 2x + y = 3, \\ 3x + 2y = 4, \\ 3z = 3. \end{cases}$$

(2 marks)

Solution.

a. Apply the inversion algorithm to $[B \mid I]$:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \overset{-R_1+R_2\to R_2}{\sim} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{-R_2+R_1\to R_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{-R_1+R_2\to R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{\frac{1}{3}R_3\to R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right].$$

Therefore

$$\boxed{B^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}}.$$

b. Multiplying by the inverse,

$$\mathbf{x} = B^{-1}\mathbf{b} = \begin{bmatrix} 2 & -1 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 6-4 \\ -9+8 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

Hence $\boxed{x = 2,\ y = -1,\ z = 1}$.

Question 6. Let

$$A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}, \qquad B = \begin{bmatrix} 3a & -2d & c+3a & b+2d \\ 3e & -2h & g+3e & f+2h \\ 3i & -2l & k+3i & j+2l \\ 3m & -2p & o+3m & n+2p \end{bmatrix}.$$

Given that $\det(A) = 7$, compute $\det(B)$. Show your work and explain how the changes from A to B affect the determinant. *(5 marks)*

Solution.

Use elementary column operations to recover A, carrying the determinant factors outside.

$$\det(B) = \begin{vmatrix} 3a & -2d & c+3a & b+2d \\ 3e & -2h & g+3e & f+2h \\ 3i & -2l & k+3i & j+2l \\ 3m & -2p & o+3m & n+2p \end{vmatrix}$$

$$\overset{-C_1+C_3\to C_3}{=} \begin{vmatrix} 3a & -2d & c & b+2d \\ 3e & -2h & g & f+2h \\ 3i & -2l & k & j+2l \\ 3m & -2p & o & n+2p \end{vmatrix}$$

$$= \begin{vmatrix} 3a & -2d & c & b \\ 3e & -2h & g & f \\ 3i & -2l & k & j \\ 3m & -2p & o & n \end{vmatrix} \quad \overset{C_2+C_4\to C_4}{}$$

$$= -6\begin{vmatrix} a & d & c & b \\ e & h & g & f \\ i & l & k & j \\ m & p & o & n \end{vmatrix} \quad \overset{\frac{1}{3}C_1\to C_1,\ -\frac{1}{2}C_2\to C_2}{}$$

$$= 6\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} \quad \overset{C_2\leftrightarrow C_4}{}$$

$$= 6\det(A) = 6(7) = \boxed{42}.$$

Adding a multiple of another column does not change the determinant. The two scalings contribute a factor of -6 outside, and the interchange changes its sign.

Question 7. Compute the following determinant. Show all your work. *(5 marks)*

$$\begin{vmatrix} 2 & 1 & 0 & 3 \\ 1 & 2 & 0 & -1 \\ 3 & 0 & 3 & 1 \\ 1 & 3 & -3 & 2 \end{vmatrix}.$$

Solution.

Expand along column 3, which contains two zeros:

$$D = 3\begin{vmatrix} 2 & 1 & 3 \\ 1 & 2 & -1 \\ 1 & 3 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 & 3 \\ 1 & 2 & -1 \\ 3 & 0 & 1 \end{vmatrix}.$$

Expand each minor along its first row:

$$\begin{vmatrix} 2 & 1 & 3 \\ 1 & 2 & -1 \\ 1 & 3 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 14.$$

$$\begin{vmatrix} 2 & 1 & 3 \\ 1 & 2 & -1 \\ 3 & 0 & 1 \end{vmatrix} = 2\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}$$

$$= -18.$$

Therefore

$$\boxed{D = 3(14) + 3(-18) = -12}.$$

Question 8. Find all matrices X, if any, such that *(5 marks)*

$$X\begin{bmatrix}1 & 0\\ 3 & 1\\ 2 & 1\end{bmatrix} = I_2.$$

Solution.

The matrix X must be 2×3. Write

$$X = \begin{bmatrix}a & b & c\\ d & e & f\end{bmatrix}.$$

Multiplying and equating corresponding entries gives

$$\begin{bmatrix}a+3b+2c & b+c\\ d+3e+2f & e+f\end{bmatrix} = I_2, \qquad \begin{cases} a+3b+2c=1,\\ b+c=0,\\ d+3e+2f=0,\\ e+f=1.\end{cases}$$

Use the variable order a, b, c, d, e, f. Gauss–Jordan elimination gives

$$\left[\begin{array}{cccccc|c} 1&3&2&0&0&0&1\\ 0&1&1&0&0&0&0\\ 0&0&0&1&3&2&0\\ 0&0&0&0&1&1&1\end{array}\right] \underset{-3R_4+R_3\to R_3}{\overset{-3R_2+R_1\to R_1}{\sim}} \left[\begin{array}{cccccc|c} 1&0&-1&0&0&0&1\\ 0&1&1&0&0&0&0\\ 0&0&0&1&0&-1&-3\\ 0&0&0&0&1&1&1\end{array}\right].$$

The matrix is in reduced row echelon form. Let $c=s$ and $f=t$. Then $a=1+s$, $b=-s$, $d=t-3$, and $e=1-t$. Therefore

$$\boxed{X=\begin{bmatrix}1+s & -s & s\\ t-3 & 1-t & t\end{bmatrix}, \qquad s,t\in\mathbb{R}}.$$

Question 9. Find two elementary matrices E_1 and E_2 that satisfy the following equation. *(4 marks)*

$$E_2E_1\begin{bmatrix}-3 & -2 & 2\\ 1 & 0 & 1\\ 0 & 1 & -2\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 1\\ 0 & 1 & -2\end{bmatrix}.$$

Solution.

Eliminate the first two entries of R_1 using R_2 and R_3:

$$\begin{bmatrix}-3 & -2 & 2\\ 1 & 0 & 1\\ 0 & 1 & -2\end{bmatrix} \underset{3R_2+R_1\to R_1}{\sim} \begin{bmatrix}0 & -2 & 5\\ 1 & 0 & 1\\ 0 & 1 & -2\end{bmatrix}$$

$$\underset{2R_3+R_1\to R_1}{\sim} \begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 1\\ 0 & 1 & -2\end{bmatrix}.$$

Apply each operation separately to I_3 to obtain

$$\boxed{E_1=\begin{bmatrix}1 & 3 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \qquad E_2=\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}}.$$

Question 10. Let A and B be $m \times n$ matrices. Prove that A and B are row equivalent if and only if there exists an invertible $m \times m$ matrix P such that $B = PA$. *(5 marks)*

Solution.

Forward direction. Suppose A and B are row equivalent. Let $E_1, \ldots, E_r$ be the elementary matrices corresponding to a sequence of row operations that transforms A into B. Then

$$B = E_r \cdots E_2 E_1 A.$$

Set $P = E_r \cdots E_2 E_1$. Each E_i is invertible, so P is invertible. Therefore $B = PA$ as required.

Reverse direction. Suppose $B = PA$ for some invertible $m \times m$ matrix P. By the Equivalence Theorem, P is a product of elementary matrices, say

$$P = F_s \cdots F_2 F_1.$$

Consequently,

$$B = F_s \cdots F_2 F_1 A.$$

Multiplication on the left by each F_i performs an elementary row operation. These operations transform A into B, so A and B are row equivalent. Both directions are proved. $\square$