DECEMBER 20, 2004

DAWSON COLLEGE
FINAL EXAM
MATHEMATICS 201-NYB-05
CALCULUS II – REGULAR

Student Name: ____________________________________

Student I.D. #: ____________________________________

Teacher: ____________________________________

Instructors: P. Birkbeck, B. Szczepara

Time: 3 Hours.

Instructions:

- **Print your name and student I.D. number in the space provided.**
- **All questions are to be answered directly on the examination paper in the provided space.**
- **Small non-programmable calculators without text storage or graphic capability are permitted.**

There are 10 problems, each is worth the same amount. Please ensure that you have a complete examination before starting.

This exam must be returned intact.

QUESTION #	MARKS
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
TOTAL	

1. Use the definition of Riemann integral to evaluate:

$$\int_0^2 \left(3x^2 - 1\right) dx \ .$$

Verify your answer by Fundamental Theorem of Calculus.

2. Sketch the region and find the area enclosed by the graphs of:

a) $y = x^2$ and $y = x^3$ b) $y = \frac{1}{x}$ and $y = \frac{1}{6}(7 - x)$.

3. Find the volume of the solid obtained when the region enclosed by the graphs of $y = 1 - x^2$ and $y = 1 - x$ is rotated about:

a) the x-axis b) the line $x = 2$.

4. Evaluate the following limits:

a) $\lim\limits_{x \to 0} \frac{1 - e^{x^2}}{3x^2}$ b) $\lim\limits_{x \to \infty} \left(1 - \frac{5}{x}\right)^x$.

5. Use integration by parts to evaluate:

a) $\int xe^{2x} dx$ b) $\int x \, \ell n \, x \, dx$.

6. Evaluate the following trigonometric integrals:

a) $\int \sin x \cos^3 x \, dx$ b) $\int \sin^2(3x) \cos^2(3x) \, dx$.

7. Use trigonometric substitution to evaluate:

a) $\int \frac{\sqrt{1 - x^2}}{x^2} dx$ b) $\int \frac{1}{x^2 \sqrt{1 + x^2}} dx$.

8. Use partial fraction to evaluate:

a) $\int \frac{5x-6}{(x-4)(x+3)}\,dx$ b) $\int \frac{x^2+2x-1}{(x^2+1)(x+1)}\,dx$.

9. Determine whether the following improper integral converges or diverges:

a) $\int_1^{\infty} \frac{2x}{3x^2+1}\,dx$ b) $\int_0^{3} \frac{1}{\sqrt{3-x}}\,dx$.

10. Determine whether the following series converges or diverges:

a) $\sum_{n=1}^{\infty} \frac{e^n}{n+e^n}$ b) $\sum_{n=1}^{\infty} \frac{5^n}{n!}$.

FINAL EXAM
NYB – REGULAR
(Solutions)

December 20, 2004

1. $$\int_0^2 \left(3x^2-1\right)dx \overset{def}{=} \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i\right)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(3\left(\tfrac{2}{n}i\right)^2-1\right)\frac{2}{n} =$$

$$= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left(\frac{12}{n^2}i^2-1\right) = \lim_{n\to\infty}\frac{2}{n}\left(\frac{12}{n^2}\sum_{i=1}^{n}n^2-\sum_{i=1}^{n}i\right) =$$

$$= \lim_{n\to\infty}\frac{2}{n}\left(\frac{12}{n^2}\frac{n\left(n+1\right)\left(2n+1\right)}{6}-n\right) = \lim_{n\to\infty}2\left(\frac{2\left(n+1\right)\left(2n+1\right)}{n^2}-1\right) =$$

$= 2\left(4-1\right)=6$. By F.T.C. $\int_0^2\left(3x^2-1\right)dx = \left(x^3-x\right)\Big|_0^2 = 8-2=6$.

2. a) $$A=\int_0^1\left(x^2-x^3\right)dx = \frac{1}{3}x^3-\frac{1}{4}x^4\Big|_0^1 = \frac{1}{3}-\frac{1}{4}=\frac{1}{12}$$

b)

$$A=\int_1^6\left(\frac{1}{6}\left(7-x\right)-\frac{1}{x}\right)dx = \frac{1}{6}\left(7x-\frac{1}{2}x^2\right)-\ell n\, x\Big|_1^6 =$$

$$=\frac{35}{12}-\ell n 6$$

3. a) $$V=\pi\int_0^1\left(\left(1-x^2\right)^2-\left(1-x\right)^2\right)dx=\pi\int_0^1\left(x^4-3x^2+2x\right)dx=$$

$$=\pi\left(\frac{1}{5}x^5-x^3+x^2\right)\Big|_0^1=\frac{\pi}{5}$$

b) $$V=2\pi\int_a^b p(x)h(x)\,dx=2\pi\int_0^1(2-x)\left(x-x^2\right)dx=$$

$$=2\pi\int_0^1\left(x^3-3x^2+2x\right)dx=2\pi\left(\tfrac{1}{4}x^4-x^3+x^2\right)\Big|_0^1=\frac{\pi}{2}$$

4. a) $$\lim_{x\to 0}\left(\frac{1-e^{x^2}}{3x^2}\right)=\lim_{x\to 0}\frac{-2xe^{x^e}}{6x}=\lim_{x\to 0}\frac{-e^{x^2}}{3}=-\frac{1}{3}$$

b) $$\lim_{x\to\infty}\left(\ell n\left(1-\frac{5}{x}\right)^x\right)=\lim_{x\to\infty}x\,\ell n\left(1-\frac{5}{x}\right)=\lim_{x\to\infty}\frac{\ell n\left(1-\frac{5}{x}\right)}{\frac{1}{x}}=$$

$$=\lim_{x\to\infty}\frac{\frac{1}{1-5/x}\left(-\frac{5}{x}\right)'}{\left(\frac{1}{x}\right)'}=\lim_{x\to\infty}\frac{-5}{1-\frac{5}{x}}=-5$$

Therefore $\lim_{x\to\infty}\left(1-\frac{5}{x}\right)^x=e^{-5}$

5. a) $$\int\underbrace{x}_{u}\underbrace{e^{2x}}_{v'}dx=uv-\int v\,du=x\left(\tfrac{1}{2}e^{2x}\right)-\int\tfrac{1}{2}e^{2x}dx=\tfrac{1}{2}xe^{2x}-\tfrac{1}{4}e^{2x}+C$$

b) $$\int\underbrace{x}_{v'}\underbrace{\ell n\,x}_{u}\,dx=uv-\int v\,du=(\ell n\,x)\left(\tfrac{1}{2}x^2\right)-\int\tfrac{1}{2}x^2\,\tfrac{1}{x}dx=\tfrac{1}{2}x^2\ell n\,x-\tfrac{1}{4}x^2+C$$

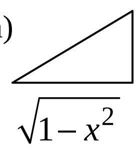
6. a) $\int \sin x \cos^3 x\, dx = \int \sin x \cos^2 x \cos x\, dx = \int (\sin x)\left(1-\sin^2 x\right)(\cos x)\, dx =$

$$\overset{u=\sin x}{=} \int u\left(1-u^2\right) du = \int \left(u-u^3\right) du = \tfrac{1}{2}\sin^2 x - \tfrac{1}{4}\sin^4 x + C$$

b) $\int \sin^2(3x)\cos^2(3x)\, dx = \int \tfrac{1}{2}(1-\cos(6x))\tfrac{1}{2}(1+\cos(6x))\, dx =$

$$= \tfrac{1}{4}\int \left(1-\cos^2(6x)\right) dx = \tfrac{1}{4}\int \sin^2(6x)\, dx = \tfrac{1}{4}\int \tfrac{1}{2}(1-\cos(12x))\, dx =$$

$$= \tfrac{1}{8}(x-\sin(12x)) + C$$

7. a) 

$\therefore\ x = \sin\alpha,\ dx = \cos\alpha\, d\alpha,\ \sqrt{1-x^2} = \cos\alpha$

$$\therefore \int \frac{\sqrt{1-x^2}}{x^2}\, dx = \int \frac{\cos\alpha}{\sin^2\alpha}(\cos\alpha)\, d\alpha = \int \cot^2\alpha\, d\alpha = \int \left(\csc^2\alpha - 1\right) d\alpha =$$

$$= -\cot\alpha - \alpha + C = -\frac{\sqrt{1-x^2}}{x} - \arcsin x + C$$

b) $\therefore\ x = \tan\alpha,\ dx = \sec^2\alpha\, d\alpha,\ \dfrac{1}{\sqrt{1+x^2}} = \cos\alpha$

$$\therefore \int \frac{1}{x^2\sqrt{1+x^2}}\, dx = \int \frac{1}{\sqrt{1+x^2}}\frac{1}{x^2}\, dx = \int (\cos\alpha)\left(\cot^2\alpha\right)\left(\sec^2\alpha\right) d\alpha =$$

$$= \int \cos\alpha \frac{\cos^2\alpha}{\sin^2\alpha}\frac{1}{\cos^2\alpha}\, d\alpha = \int \frac{\cos\alpha}{\sin^2\alpha}\, d\alpha \overset{u=\sin\alpha}{=} \int u^{-2} du = -\frac{1}{u} + C =$$

$$= -\frac{1}{\sin\alpha} + C = -\frac{\sqrt{1+x^2}}{x} + C$$

8. a) $\dfrac{5x-6}{(x-4)(x+3)}=\dfrac{A}{x-4}+\dfrac{B}{x+3}=\dfrac{A(x+3)+B(x-4)}{(x-4)(x+3)}=\dfrac{(A+B)x+(3A-4B)}{(x-4)(x+3)}$

$$\therefore \left.\begin{array}{r} A+\ \ B=\ \ 5 \\ 3A-4B=-6 \end{array}\right\} \Rightarrow A=2 \quad and \quad B=3$$

$$\therefore \int \frac{5x-6}{(x-4)(x+3)}\,dx = 2\ell n|x-4|+3\ell n|x+3|+C$$

b) $\dfrac{x^2+2x-1}{\left(x^2+1\right)(x+1)}=\dfrac{Ax+B}{x^2+1}+\dfrac{C}{x+1}=\dfrac{(Ax+B)(x+1)+C\left(x^2+1\right)}{\left(x^2+1\right)(x+1)}=$

$$=\frac{(A+C)x^2+(A+B)x+(B+C)}{\left(x^2+1\right)(x+1)} \qquad \therefore \left.\begin{array}{r} A+C=\ \ 1 \\ A+B=\ \ 2 \\ B+C=-1 \end{array}\right\} \Rightarrow \begin{array}{r} A=\ \ 2 \\ B=\ \ 0 \\ C=-1 \end{array}$$

$$\therefore \frac{x^2+2x-1}{\left(x^2+1\right)(x+1)}dx=\int\left(\frac{2x}{x^2+1}\cdots\frac{1}{x+1}\right)dx=\ell n\left(x^2+1\right)-\ell n|x+1|+C$$

9. a) $\displaystyle\int_1^{\infty}\frac{2x}{3x^2+1}\,dx=\lim_{b\to\infty}\int_1^{b}\frac{2x}{3x^2+1}\,dx=\lim_{b\to\infty}\frac{1}{3}\,\ell n\left(3x^2+1\right)\Big|_1^b=$

$$=\lim_{b\to\infty}\frac{1}{3}\left(\ell n\left(3b^2+1\right)-\ell n\ 4\right)=\infty.\ \therefore\ \underline{\text{It diverges}}.$$

b) $\displaystyle\int_0^{3}\frac{1}{\sqrt{3-x}}\,dx=\lim_{b\to 3^-}\int_0^{b}\frac{1}{\sqrt{3-x}}\,dx=\lim_{b\to 3^-}\left(-2\sqrt{3-x}\right)\Big|_0^b=$

$$=\lim_{b\to 3^-}\left(-2\sqrt{3-b}+2\sqrt{3}\right)=2\sqrt{3}.\ \therefore\ \underline{\text{It converges}}.$$

10. a) $\lim_{n\to\infty} \frac{e^n}{n+e^n} = \lim_{n\to\infty} \frac{e^n}{1+e^n} = \lim_{n\to\infty} \frac{e^n}{e^n} = 1 \neq 0$

$\therefore \sum_{n=1}^{\infty} \frac{e^n}{n+e^n}$ diverges.

b) (Ratio Test): $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left(\frac{5^{n+1}}{(n+1)!} \frac{n!}{5^n} \right) =$

$= \lim_{n\to\infty} \frac{5}{n+1} = 0 < 1$. Therefore the series