Dawson College: Calculus II: 201-NYB-05-S2: Summer 2008

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Test 2

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Find the average of the function $f(x) = \frac{x(x^2-1)}{x^2+1}$ over the interval $[-1, 1]$.

$$\text{average} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{1-(-1)}\int_{-1}^{1} f(x)\,dx$$

$$= \frac{1}{2}[0] \text{ since } f(x) \text{ is odd.}$$

$$= 0$$

$$f(-x) = \frac{-x((-x)^2-1)}{(-x)^2+1}$$

$$= \frac{-x(x^2-1)}{x^2+1}$$

$$= -f(x)$$

$\therefore f(x)$ is odd

Question 2. *(5 marks)* Evaluate the following definite integral:

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\,dx \overset{①,②}{=} \int_0^1 \frac{du}{1+u^2} = \Big[\arctan u\Big]_0^1$$

$$= \arctan 1$$

$$= \pi/4$$

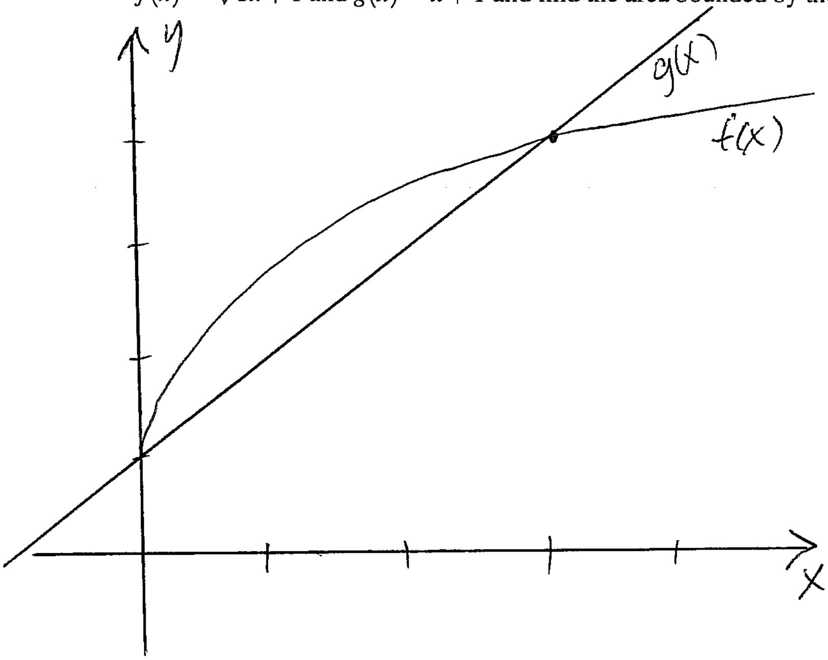
① $u = \sin x$

② $du = \cos x\,dx$

$u(0) = \sin 0 = 0$

$u(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$

Question 3. *(2 marks for sketch and 3 marks for area)* Sketch the graph of the two following algebraic functions $f(x) = \sqrt{3x} + 1$ and $g(x) = x + 1$ and find the area bounded by the two functions.



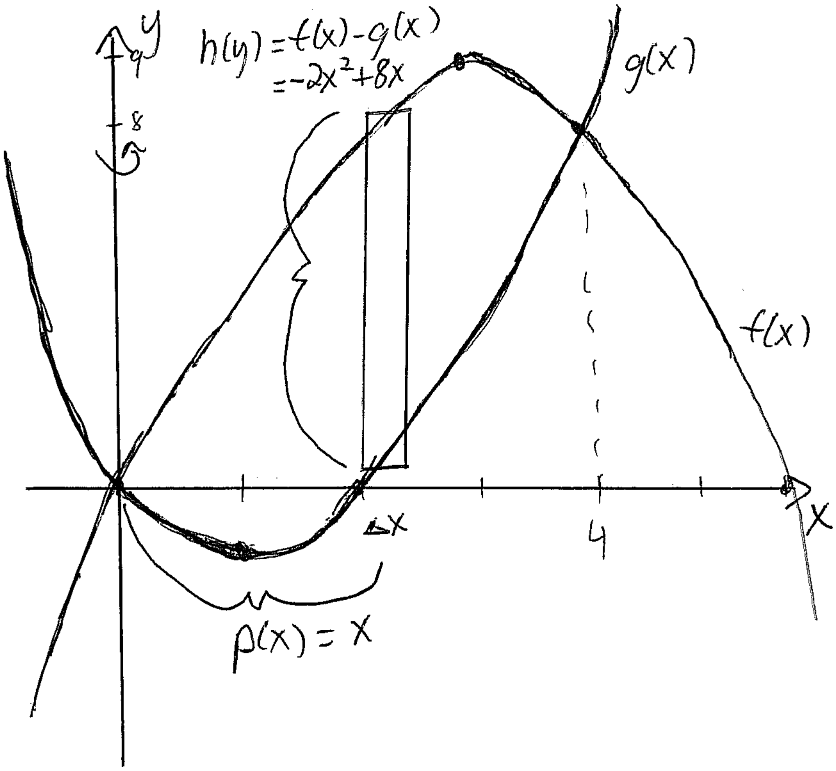
Lets find the intersection of both curves.

$$f(x) = g(x)$$
$$\sqrt{3x} + 1 = x + 1$$
$$\sqrt{3x} = x \quad \text{sbs}$$
$$3x = x^2$$
$$0 = x^2 - 3x$$
$$0 = x(x-3)$$

$\therefore$ intersection at $x=0$, $x=3$

$$\text{Area} = \int_0^3 f(x) - g(x)\,dx$$
$$= \int_0^3 \sqrt{3x} + 1 - (x+1)\,dx$$
$$= \int_0^3 \sqrt{3x} - x\,dx$$
$$= \left[\frac{2(3x)^{3/2}}{9} - \frac{x^2}{2}\right]_0^3$$
$$= \frac{2(3(3))^{3/2}}{9} - \frac{3^2}{2}$$
$$= \frac{54}{9} - \frac{9}{2}$$
$$= 6 - \frac{9}{2} = \frac{3}{2}$$

Question 4. *(5 marks)* Find the volume of the solid of revolution generated by the bounded region of the functions $f(x) = -x^2+6x$ and $g(x) = x^2-2x$ revolved about the y-axis.



Lets find the intersection of both curves.

$$f(x) = g(x)$$
$$-x^2+6x = x^2-2x$$
$$0 = 2x^2-8x$$
$$0 = 2x(x-4)$$

$\therefore$ intersection at $x=0$ and $x=4$

<u>representative element:</u>

$$\Delta V = 2\pi p(x) h(x) \Delta x$$
$$= 2\pi x(-2x^2+8x)\Delta x$$
$$= 2\pi(-2x^3+8x^2)\Delta x$$

$$V = \int_0^4 2\pi(-2x^3+8x^2)dx$$

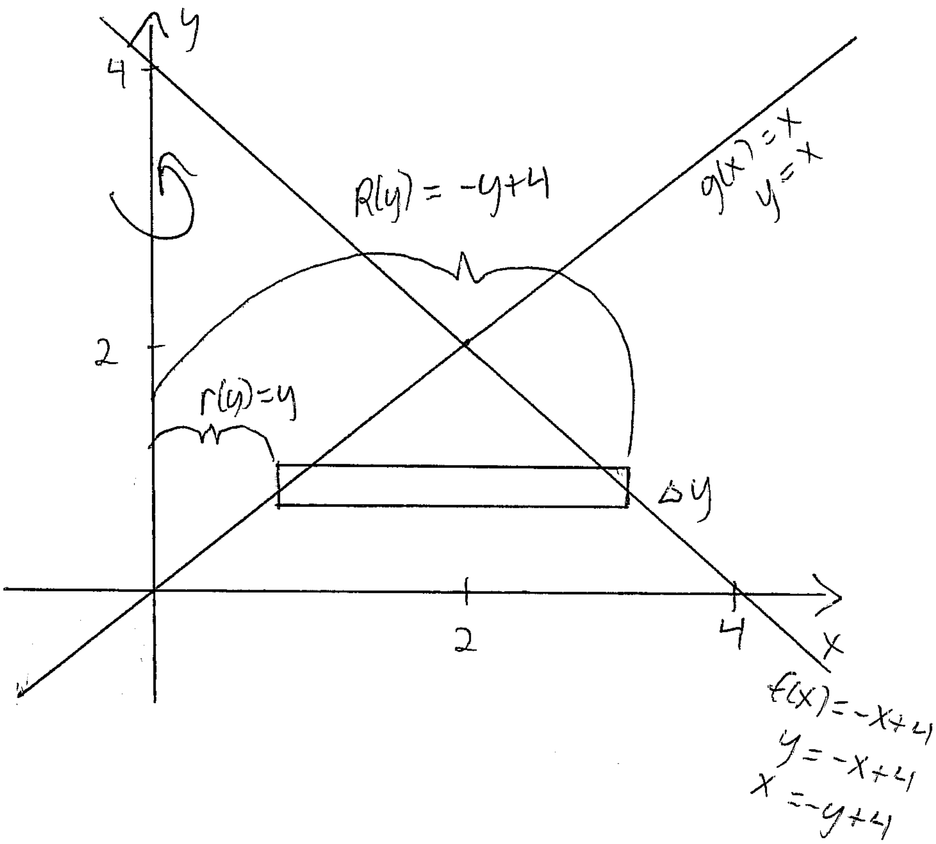
$$= 2\pi\left[\frac{-2x^4}{4} + \frac{8x^3}{3}\right]_0^4$$

$$= 2\pi\left[\frac{-2(4)^{4^3}}{4} + \frac{8(4)^3}{3}\right]$$

$$= 2\pi\left[-128 + \frac{512}{3}\right]$$

$$= \frac{256}{3}\pi$$

Question 5. *(5 marks)* Find the volume of the solid of revolution generated by the bounded region of the functions $f(x) = -x+4$, $g(x) = x$ and $y = 0$ revolved about y-axis.



Lets find the intersection between the two curves

$$f(x) = g(x)$$
$$-x+4 = x$$
$$4 = 2x$$
$$2 = x$$

representative element:

$$\Delta V = \pi\left[(R(y))^2 - (r(y))^2\right]\Delta y$$
$$= \pi\left[(4-y)^2 - y^2\right]\Delta y$$
$$= \pi\left[16 - 8y + y^2 - y^2\right]\Delta y$$
$$= \pi\left[16 - 8y\right]$$

$$V = \int_0^2 \pi[16-8y]\,dy$$
$$= \pi\left[16y - 4y^2\right]_0^2$$
$$= \pi\left[16(2) - 4(2)^2\right]$$
$$= \pi\left[32 - 16\right]$$
$$= 16\pi$$

Question 6. *(5 marks)* Find the arc length of the graph of the function $y=\frac{3}{2}x^{2/3}$ over the interval $[1,8]$.

$$S=\int_1^8 \sqrt{1+(f'(x))^2}\,dx \qquad f'(x)=x^{-1/3}$$

$$=\int_1^8 \sqrt{1+(x^{-2/3})}\,dx$$

$$u=x^{2/3}+1$$

$$du=\frac{2dx}{3x^{1/3}}$$

$$\frac{3du}{2}=\frac{dx}{x^{1/3}}$$

$$u(0)=1$$

$$u(8)=5$$

$$=\int_1^8 \sqrt{\frac{x^{2/3}+1}{x^{2/3}}}\,dx$$

$$=\int_1^8 \sqrt{x^{2/3}+1}\;\frac{dx}{x^{1/3}}$$

$$=\int_1^5 \sqrt{u}\;\frac{3du}{2}$$

$$=\frac{3}{2}\int_1^5 \sqrt{u}\,du$$

$$=\frac{\cancel{3}}{\cancel{2}}\left[\frac{\cancel{2}u^{3/2}}{\cancel{3}}\right]_1^5$$

$$=5^{3/2}-1^{3/2}$$

$$=\sqrt{125}-1$$

Question 7. *(5 marks)* Use the Trapezoidal Rule with $n = 4$ to approximate the value of the definite integral and compare your answer to the exact value of the definite integral. *(i.e. calculate the definite integral using the Fundamental Theorem of Calculus.)*

$$\int_0^2 xe^{x^2}\,dx \overset{①,②}{=} \int_0^4 e^u \frac{du}{2} = \frac{1}{2}\left[e^u\right]_0^4 = \frac{e^4-1}{2} \doteq 26.7991$$

① $u = x^2$

$du = 2x\,dx$

② $\frac{du}{2} = x\,dx$

$u(0) = 0^2 = 0$

$u(2) = 2^2 = 4$

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$$

$$x_i = a + i\Delta x$$

$$x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1,\ x_3 = 3/2,\ x_4 = 2$$

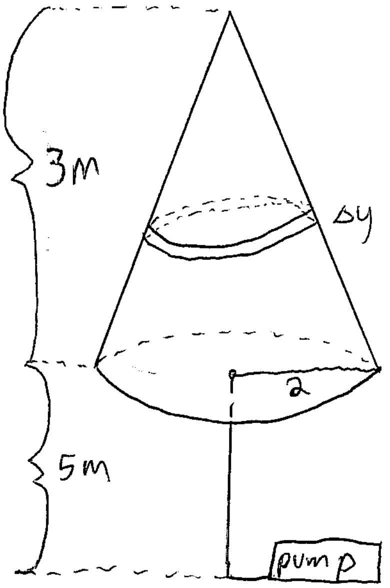
$$\approx \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$= \frac{2-0}{2(4)}\left[0e^{0^2} + 2\left(\tfrac{1}{2}\right)e^{(\frac{1}{2})^2} + 2(1)e^{1^2} + 2\left(\tfrac{3}{2}\right)e^{(3/2)^2} + 2e^{2^2}\right]$$

$$= \frac{\cancel{2}}{\cancel{2}(4)}\left[0 + e^{1/4} + 2e^1 + 3e^{9/4} + 2e^4\right]$$

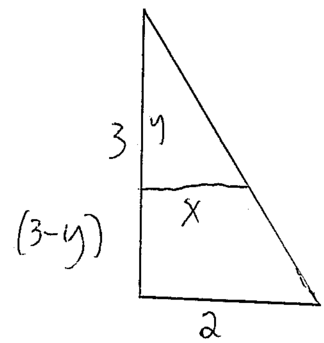
$$= 36.0950$$

Question 8. *(5 marks)* A conical tank with its tip pointing upwards is filled by a pump which is located 5m below the tank. If the fluid that has a density of $\rho = 2000\frac{\text{kg}}{\text{m}^3}$ and the tank is 4m across at the bottom and 3m in height, how much work is required to fill the tank?



Volume of slice: $\Delta V = \pi x^2 \Delta y$

Lets obtain the volume of the slice only in terms of y



$$\frac{x}{2} = \frac{(3-y)}{3}$$

$$x = \frac{2(3-y)}{3}$$

$$\therefore \Delta V = \pi \frac{4}{9}(3-y)^2 \Delta y$$

mass of slice! $\Delta m = \rho \Delta V = \rho \pi \frac{4}{9}(3-y)^2 \Delta y$

force excerted by slice: $\Delta F = \Delta m g = \rho g \pi \frac{4}{9}(3-y)^2 \Delta y$

distance from pump to slice: $d = y + 5$

work to move slide ; $\Delta W = \Delta F d = \rho g \pi \frac{4}{9}(3-y)^2 (y+5) \Delta y$

$$W = \int_0^3 \rho g \pi \frac{4}{9}(3-y)^2 (y+5)\,dy$$

$$= \rho g \pi \frac{4}{9} \int_0^3 (9 - 6y + y^2)(y+5)\,dy$$

$$= \rho g \pi \frac{4}{9} \int_0^3 9y - 6y^2 + y^3 + 45 - 30y + 5y^2\,dy$$

$$= \rho g \pi \frac{4}{9} \left[-\frac{21y^2}{2} - \frac{y^3}{3} + \frac{y^4}{4} + 45y \right]_0^3$$

$$= \rho g \pi \frac{4}{9} \left(\frac{69}{2}\right) \text{ N}\cdot\text{m}$$

$$\doteq 944\,153.3 \text{ N}\cdot\text{m}$$

Question 9. *(5 marks)* Find the 'c' value(s) guaranteed by the Mean Value Theorem for Integrals for the function $f(x) = \tan x$ over $[\frac{\pi}{6}, \frac{\pi}{3}]$.

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

$$\int_{\pi/6}^{\pi/3} \tan x\,dx = \tan c\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$\Big[-\ln|\cos x|\Big]_{\pi/6}^{\pi/3} = \tan c\left(\frac{\pi}{6}\right)$$

$$\ln|\cos\tfrac{\pi}{6}| - \ln|\cos\tfrac{\pi}{3}| = \tan c\left(\frac{\pi}{6}\right)$$

$$\ln\left(\frac{\sqrt{3}}{2}\right) - \ln\left(\frac{1}{2}\right) = \tan c\left(\frac{\pi}{6}\right)$$

$$\ln\left(\sqrt{3}\right) = \tan c\left(\frac{\pi}{6}\right)$$

$$\frac{6\ln(\sqrt{3})}{\pi} = \tan c$$

$$\tan^{-1}\left(\frac{6\ln\sqrt{3}}{\pi}\right) = c$$

$$c \doteq 0.8094\ \text{rad}$$

$$c \doteq 46.37^\circ$$

Bonus Question. *(5 marks)*

Prove one of the following statement:

- If $f(x)$ is an even function then

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

- The formula of the arc length of a function. (in class notes)

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

substitution for first integral.

$-u = x$

$-du = dx$

$u(-a) = a$

$u(0) = 0$

$$= \int_a^0 f(-u)\,(-du) + \int_0^a f(x)\,dx$$

since $f(x)$ is even

$$= -\int_a^0 f(u)\,du + \int_0^a f(x)\,dx$$

$$= \int_0^a f(u)\,du + \int_0^a f(x)\,dx$$

$$= 2\int_0^a f(x)\,du$$