Dawson College: **Calculus II**: 201-NYB-05-S2: Summer 2008

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Test 2

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Find the average of the function $f(x) = \frac{x(x^2-2)}{x^2+2}$ over the interval $[-2, 2]$.

$$\text{average} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{2-(2)}\int_{-2}^{2} f(x)\,dx$$

$$= \frac{1}{4}[0] \text{ since } f(x) \text{ is odd}$$

$$= 0$$

$$f(-x) = \frac{-x((-x)^2-2)}{(-x)^2+2}$$

$$= -\frac{x(x^2-2)}{x^2+2}$$

$$= -f(x)$$

$\therefore f(x)$ is odd

Question 2. *(5 marks)* Evaluate the following definite integral:

$$\int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx \overset{①,②}{=} \int_0^{-1} \frac{-du}{1+u^2} = \left[-\arctan u\right]_0^{-1}$$

① $u = \cos x$

$du = -\sin x\,dx$

② $-du = \sin x\,dx$

$u(\pi/2) = \cos \pi/2 = 0$

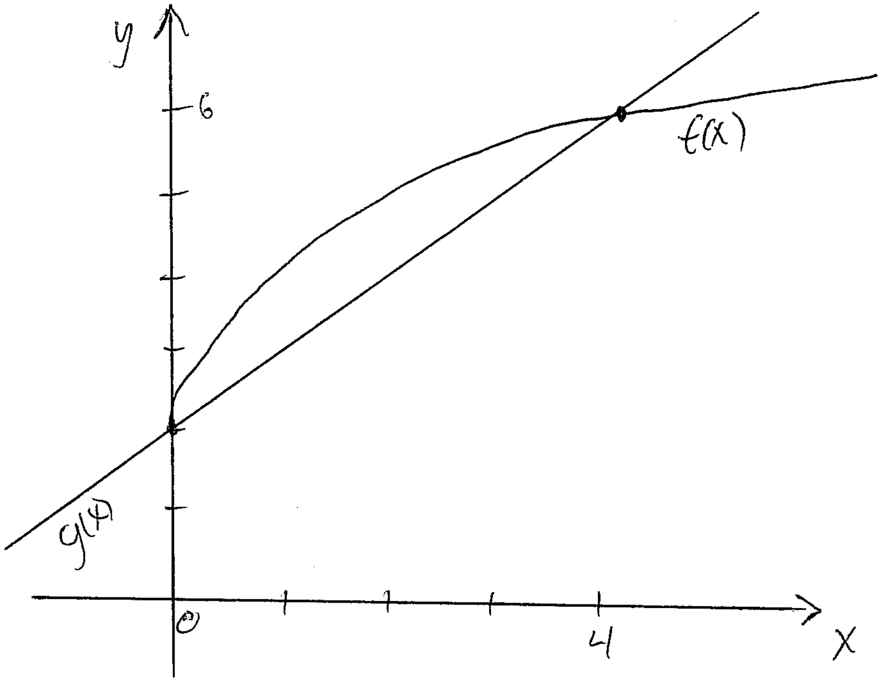
$u(\pi) = \cos \pi = -1$

$$= -\arctan -1 + \arctan 0$$

$$= -\left(\frac{-\pi}{4}\right) + 0$$

$$= \frac{\pi}{4}$$

Question 3. *(2 marks for sketch and 3 marks for area)* Sketch the graph of the two following algebraic functions $f(x) = \sqrt{4x} + 2$ and $g(x) = x + 2$ and find the area bounded by the two functions.

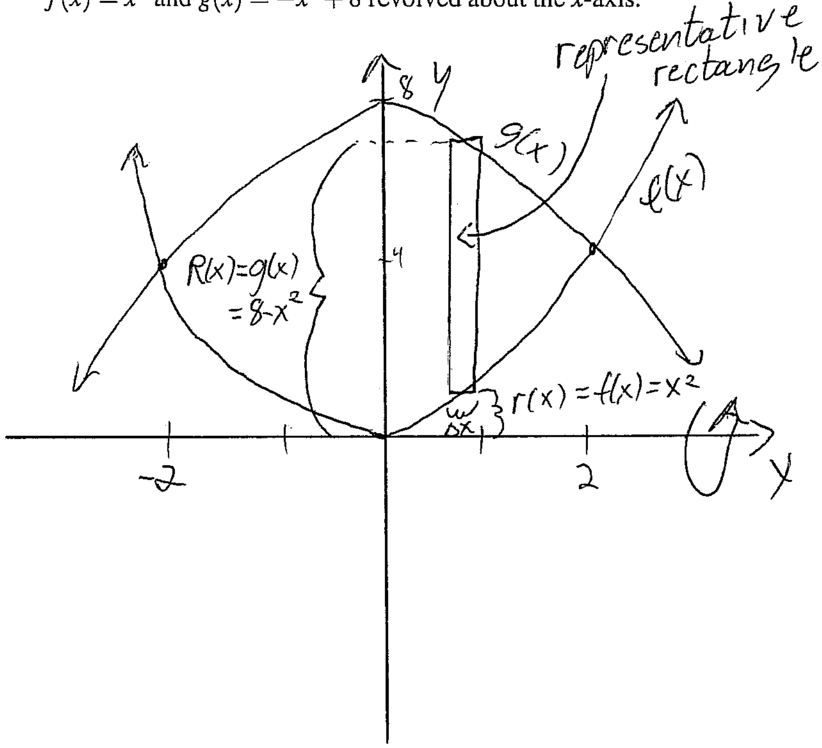


Lets find the intersection of the two curves

$$f(x) = g(x)$$
$$\sqrt{4x} + 2 = x + 2$$
$$\sqrt{4x} = x \quad \text{sbs}$$
$$4x = x^2$$
$$0 = x^2 - 4x$$
$$0 = x(x-4)$$
at $x=0$ and $x=4$

$$\text{Area} = \int_0^4 f(x) - g(x)\,dx$$
$$= \int_0^4 \sqrt{4x} + 2 - (x+2)\,dx$$
$$= \int_0^4 \sqrt{4x} - x\,dx$$
$$= \left[\frac{2(4x)^{3/2}}{3\cdot 4} - \frac{x^2}{2}\right]_0^4$$
$$= \left[\frac{\cancel{2}(4\cdot 4)^{3/2}}{3\cdot \cancel{4}2} - \frac{4^2}{2}\right]$$
$$= \frac{32}{3} - 8 = \frac{8}{3}$$

Question 4. *(5 marks)* Find the volume of the solid of revolution generated by the bounded region of the functions $f(x) = x^2$ and $g(x) = -x^2 + 8$ revolved about the x-axis.



Lets find the intersetien of the two curves

$$f(x) = g(x)$$
$$x^2 = -x^2 + 8$$
$$2x^2 = 8$$
$$x^2 = 4$$
$$x = \pm 2$$

representative element:

$$\Delta V = \pi\left[(R(x))^2 - (r(x))^2\right]\Delta x$$
$$= \pi\left[(-x^2+8)^2 - (x^2)^2\right]\Delta x$$
$$= \pi\left[x^4 - 16x^2 + 64 - x^4\right]\Delta x$$
$$= \pi\left[64 - 16x^2\right]\Delta x$$

$$V = \int_{-2}^{2} \pi\left[64 - 16x^2\right]dx$$

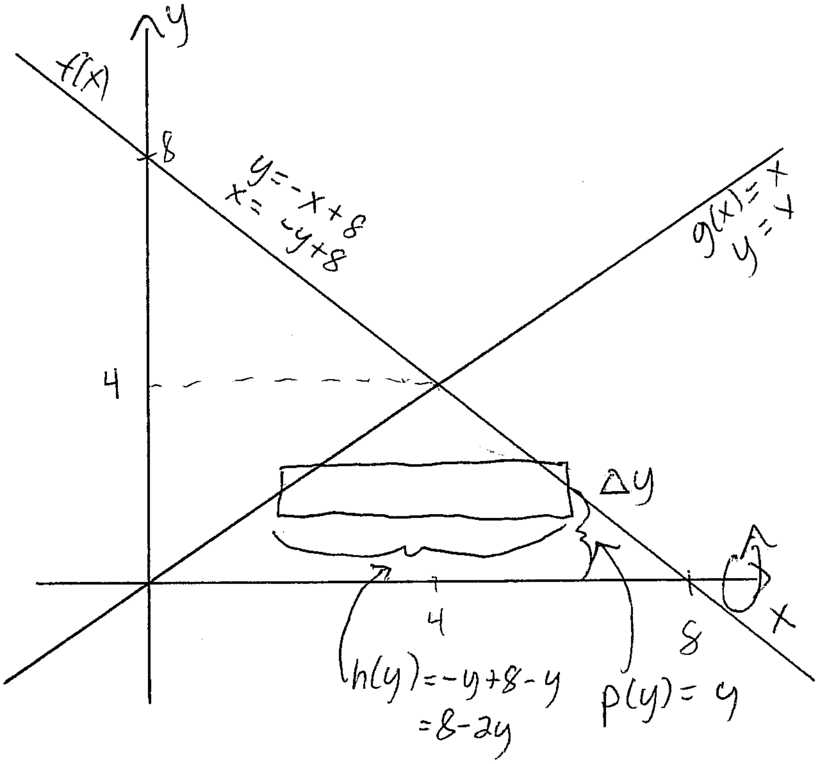
$$= \left[\pi\left[64x - \frac{16x^3}{3}\right]\right]_{-2}^{2}$$

$$= \pi\left[64(2) - \frac{16(2)^3}{3}\right] - \pi\left[64(-2) - \frac{16(-2)^3}{3}\right]$$

$$= \pi\left[256 - \frac{256}{3}\right]$$

$$= \frac{512\pi}{3}$$

Question 5. *(5 marks)* Find the volume of the solid of revolution generated by the bounded region of the functions $f(x) = -x + 8$, $g(x) = x$ and $y = 0$ revolved about x-axis.



Lets determine where the two lines intersect

$$f(x) = g(x)$$
$$-x + 8 = x$$
$$8 = 2x$$
$$4 = x$$

<u>representative element:</u>

$$\Delta V = 2\pi p(y) h(y) \Delta y$$
$$= 2\pi y (8 - 2y) \Delta y$$

$$V = \int_0^4 2\pi y(8-2y) dy$$

$$= 2\pi \left[\frac{8y^2}{2} - \frac{2y^3}{3} \right]_0^4$$

$$= 2\pi \left[\frac{\cancel{8}4(4)^2}{\cancel{2}} - \frac{2(4)^3}{3} \right] = 2\pi \left[64 - \frac{128}{3} \right]$$

$$= \frac{128\pi}{3}$$

Question 6. *(5 marks)* Find the arc length of the graph of the function $f(x) = \frac{x^3}{6} + \frac{1}{2x}$ over the interval $[1,2]$.

$$S = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

Lets determine the derivative and simplify / factor the radicand.

$$f'(x) = \frac{3x^2}{6} - \frac{1}{2x^2} = \frac{x^2}{2} - \frac{1}{2x^2}$$

radicand:

$$1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2 = 1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}$$

$$= \frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}$$

$$= \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2$$

$$\therefore S = \int_1^2 \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2}\,dx$$

$$= \int_1^2 \frac{x^2}{2} + \frac{1}{2x^2}\,dx$$

$$= \left[\frac{x^3}{6} - \frac{1}{2x^1}\right]_1^2$$

$$= \frac{2^3}{6} - \frac{1}{2(2)} - \frac{1}{6} + \frac{1}{2}$$

$$= \frac{8}{6} - \frac{1}{4} - \frac{1}{6} + \frac{1}{2}$$

$$= \frac{17}{12}$$

Question 7. *(5 marks)* Use the Simpson's Rule with $n = 4$ to approximate the value of the definite integral and compare your answer to the exact value of the definite integral. *(i.e. calculate the definite integral using the Fundamental Theorem of Calculus.)*

$$\int_0^2 xe^{x^2}\,dx \overset{①,②}{=} \int_0^4 e^u \frac{du}{2} = \left[\frac{e^u}{2}\right]_0^4 = \frac{e^4}{2} - \frac{e^0}{2} \doteq 26.7991$$

① $u = x^2$

$du = 2x\,dx$

② $\frac{du}{2} = x\,dx$

$u(0) = 0^2 = 0$

$u(2) = 2^2 = 4$

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$$

$x_i = a + i\Delta x$

$x_0 = 0,\quad x_1 = \frac{1}{2},\quad x_2 = 1,\quad x_3 = \frac{3}{2},\quad x_4 = 2$

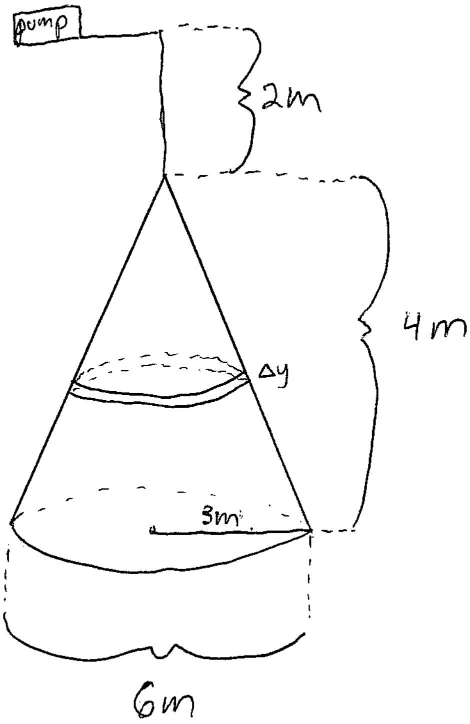
$$\approx \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$

$$= \frac{2-0}{3(4)}\left[0e^{0^2} + 4\left(\tfrac{1}{2}\right)e^{\left(\frac{1}{2}\right)^2} + 2(1)e^{1^2} + 4\left(\tfrac{3}{2}\right)e^{\left(\frac{3}{2}\right)^2} + 2e^{2^2}\right]$$

$$= \frac{1}{6}\left[0 + 2e^{1/4} + 2e + 6e^{9/4} + 2e^4\right]$$

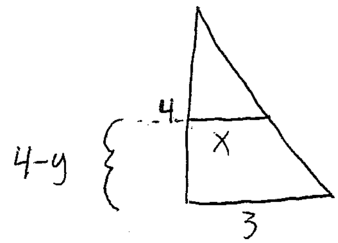
$\doteq 29.0212$

Question 8. *(5 marks)* A conical tank with its tip pointing upwards is emptied by a pump which is located 2m above the tank. If the fluid that has a density of $\rho = 2000\frac{\text{kg}}{\text{m}^3}$ and the tank is 6m across at the bottom and 4m in height, how much work is required to empty the tank?



volume of slice: $\Delta V = \pi (X)^2 \Delta y$

now the volume of the slice all with respelt to y



$\therefore \Delta V = \pi\left(\frac{3}{4}(4-y)\right)^2 \Delta y$

$$\frac{4-y}{X} = \frac{4}{3}$$

$$\frac{3(4-y)}{4} = X$$

mass of slice: $\Delta m = \rho \pi \frac{9}{16}(4-y)^2 \Delta y$

force excerted by slice: $\Delta F = \Delta m g = \rho g \frac{9\pi}{16}(4-y)^2 \Delta y$

distance of the slice to pump: $d = 6-y$

work: $\Delta W = \Delta F d = \rho g \frac{9\pi}{16}(4-y)^2(6-y)\Delta y$

$$\text{work} = \int_0^4 \rho g \frac{9\pi}{16}(4-y)^2(6-y)\,dy$$

$$= \frac{9\pi\rho g}{16}\int_0^4 (16-8y+y^2)(6-y)\,dy = \frac{9\pi\rho g}{16}\int_0^4 (96-48y+6y^2-16y+8y^2 \; y^3)\,dy$$

$$= \frac{9\pi\rho g}{16}\left[96y - 32y^2 + 14\frac{y^3}{3} - \frac{y^4}{4}\right]_0^4$$

$$= \frac{9}{16}\pi\rho g\left[96(4) - 32(4)^2 + \frac{14(4)^3}{3} - \frac{4^4}{4}\right] = \frac{9}{16}\pi\rho g\left(\frac{320}{3}\right) \text{N}\cdot\text{m} \doteq 3\,694\,512 \text{ N}\cdot\text{m}$$

Question 9. *(5 marks)* Find the 'c' value(s) guaranteed by the Mean Value Theorem for Integrals for the function $f(x) = \cot x$ over $[\frac{\pi}{6}, \frac{\pi}{4}]$.

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

$$\int_{\pi/6}^{\pi/4} \cot x\,dx = \cot c\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$\Big[\ln|\sin x|\Big]_{\pi/6}^{\pi/4} = \cot c\left(\frac{\pi}{12}\right)$$

$$\ln\sin\frac{\pi}{4} - \ln\sin\frac{\pi}{6} = \cot c\left(\frac{\pi}{12}\right)$$

$$\ln\frac{1}{\sqrt{2}} - \ln\frac{1}{2} = \cot c\left(\frac{\pi}{12}\right)$$

$$\frac{12}{\pi}\ln\frac{2}{\sqrt{2}} = \cot c$$

$$\frac{12}{\pi}\ln\frac{2}{\sqrt{2}} = \frac{1}{\tan c}$$

$$\tan c = \frac{\pi}{12\ln 2/\sqrt{2}}$$

$$c = \tan^{-1}\left[\frac{\pi}{12\ln 2/\sqrt{2}}\right]$$

$$\doteq 0.6469 \text{ rad}$$

$$\doteq 37^{\circ}$$

Bonus Question. *(5 marks)*
Prove one of the following statement:

- If $f(x)$ is an even function then
$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$
- The formula of the arc length of a function. (given in class)

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

change of variable $-x=u$

$$= \int_a^0 f(-u)\,(-du) + \int_0^a f(x)\,dx$$

$$= -\int_a^0 f(-u)\,du + \int_0^a f(x)\,dx$$

$$= -\int_a^0 f(u)\,du + \int_0^a f(x)\,dx \quad \text{since } f(x) \text{ is even}$$

$$= \int_0^a f(u)\,du + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$$