

Test 3

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. (5 marks) Integrate the following indefinite integral:

$$\begin{aligned}
 \int_{\pi/3}^{\pi/4} x \csc^2 x \, dx &= \left[uv \right]_{\pi/3}^{\pi/4} - \int_{\pi/3}^{\pi/4} v \, du & u=x & \quad dw=dx \\
 & & v=-\cot x & \quad dv = \csc^2 x \, dx \\
 &= \left[-x \cot x \right]_{\pi/3}^{\pi/4} + \int_{\pi/3}^{\pi/4} \cot x \, dx \\
 &= -\frac{\pi}{4} \cot \frac{\pi}{4} + \frac{\pi}{3} \cot \frac{\pi}{3} + \left[\ln |\sin x| \right]_{\pi/3}^{\pi/4} \\
 &= -\frac{\pi}{4} + \frac{\pi}{\sqrt{3}(3)} + \ln |\sin \frac{\pi}{4}| - \ln |\sin \frac{\pi}{3}| \\
 &= \frac{-\pi}{4} + \frac{\pi}{\sqrt{3}(3)} + \ln \left| \frac{1}{\sqrt{2}} \right| - \ln \left| \frac{\sqrt{3}}{2} \right|
 \end{aligned}$$

Question 2. (5 marks) Integrate the following indefinite integral:

$$\begin{aligned}
 \int \sin^4 x \, dx &= \int (\sin^2 x)^2 \, dx \\
 &= \int \left(\frac{1 - \cos 2x}{2} \right)^2 \, dx \\
 &= \frac{1}{4} \int (1 - \cos 2x)^2 \, dx \\
 &= \frac{1}{4} \int 1 - 2\cos 2x + \cos^2 2x \, dx \\
 &= \frac{1}{4} \int 1 - 2\cos 2x + \frac{1}{2} + \frac{\cos 4x}{2} \, dx \\
 &= \frac{1}{4} \left[x - \sin 2x + \frac{x}{2} + \frac{\sin 4x}{8} \right] + C \\
 &= \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C
 \end{aligned}$$

Question 3. (5 marks) Integrate the following indefinite integral:

$$\int \frac{\sqrt{4-x^2}}{x} dx$$

$$x = 2 \sin \theta$$

$$dx = 2 \cos \theta d\theta$$

$$= \int \frac{\sqrt{4-4\sin^2\theta} \cdot 2\cos\theta d\theta}{2\sin\theta}$$

$$= \int \frac{\sqrt{4(1-\sin^2\theta)} \cos\theta d\theta}{\sin\theta}$$

$$= \int \frac{\sqrt{4\cos^2\theta} \cos\theta d\theta}{\sin\theta}$$

$$= 2 \int \frac{\cos^2\theta}{\sin\theta} d\theta$$

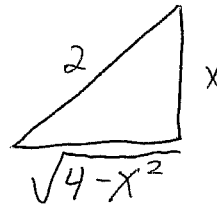
$$= 2 \int \frac{1-\sin^2\theta}{\sin\theta} d\theta$$

$$= 2 \int \csc\theta - \sin\theta d\theta$$

$$= 2 \left[-\ln|\csc\theta| + \cot\theta + \cos\theta \right] + C$$

$$= -2 \ln \left| \frac{2}{x} + \frac{\sqrt{4-x^2}}{x} \right| + \sqrt{4-x^2} + C$$

$$\frac{x}{2} = \sin\theta$$



$$\cos\theta = \frac{a}{h} = \frac{\sqrt{4-x^2}}{2}$$

$$\cot\theta = \frac{a}{o} = \frac{\sqrt{4-x^2}}{x}$$

$$\csc\theta = \frac{2}{x}$$

Question 4. (5 marks) Integrate the following indefinite integral:

$$\int \frac{x-1}{x(x+1)^2} dx$$

lets decompose using partial fractions

$$\frac{x-1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x-1 = A(x+1)^2 + Bx(x+1) + Cx$$

Let $x=0$

$$-1 = A + B(0)(0+1) + C(0)$$

$$-1 = A$$

Let $x=-1$

$$-1-1 = A(-1+1)^2 + B(-1)(-1+1) + C$$

$$2 = C$$

Let $x=1$

$$1-1 = -1(1+1)^2 + B(1)(1+1) + 2(1)$$

$$0 = -2^2 + 2B + 2$$

$$1 = B$$

$$\begin{aligned} \int \frac{x-1}{x(x+1)^2} dx &= \int \frac{-1}{x} + \frac{2}{(x+1)^2} + \frac{1}{x+1} dx \\ &= -\ln|x| - \frac{2}{x+1} + \ln|x+1| + C \end{aligned}$$

Question 5. (5 marks) Evaluate the limit, using L'Hôpital's Rule if necessary.

$$\lim_{x \rightarrow \infty} \frac{x^2}{e^x} \text{ has IF } \frac{\infty}{\infty} \text{ so we apply L'Hôpital's Rule}$$

$$= \lim_{x \rightarrow \infty} \frac{2x}{e^x} \text{ has IF } \frac{\infty}{\infty} \text{ so we apply L'Hôpital's Rule}$$

$$= \lim_{x \rightarrow \infty} \frac{2}{e^x}$$

$$= 0$$

Question 6. (5 marks) Solve the following improper integral:

$$\int_{-\infty}^0 x e^x dx$$

$$= \lim_{a \rightarrow -\infty} \int_a^0 x e^x dx$$

$$= \lim_{a \rightarrow -\infty} \left[[uv]_a^0 - \int_a^0 v du \right]$$

$$= \lim_{a \rightarrow -\infty} \left[[x e^x]_a^0 - \int_a^0 e^x dx \right]$$

$$= \lim_{a \rightarrow -\infty} 0e^0 - ae^a - [e^x]_a^0$$

$$= \lim_{a \rightarrow -\infty} -ae^a - e^0 + e^{a^0}$$

$$= -1 - \lim_{a \rightarrow -\infty} ae^a \quad \text{has IF } -\infty 0$$

$$= -1 - \lim_{a \rightarrow -\infty} \frac{a}{e^{-a}} \quad \text{has IF } \frac{-\infty}{\infty} \quad \text{so we apply L'Hopital's Rule}$$

$$= -1 - \lim_{a \rightarrow -\infty} \frac{1}{e^{-a}}$$

$$= -1$$

$$u = x$$

$$v = e^x$$

$$du = dx$$

$$dv = e^x dx$$

Question 7. (5 marks) Solve the following improper integral:

$$\int_0^{11} \frac{7}{(11-x)^3} dx \quad \text{has an infinite discontinuity at } x=11.$$

$$= \lim_{b \rightarrow 11} \int_0^b \frac{7}{(11-x)^3} dx$$

$$= \lim_{b \rightarrow 11} \left[\frac{-7}{2(11-x)^2} \right]_0^b$$

$$= \lim_{b \rightarrow 11} \left[\frac{-7}{2(11-b)^2} + \frac{7}{2(11-0)^2} \right]$$

\therefore the integral diverges.

Question 8. (5 marks) Integrate the following indefinite integral:

$$\int \frac{1}{1-e^{3x}} dx = \int \frac{1}{1-e^{3x}} \left(\frac{e^{-3x}}{e^{-3x}} \right) dx$$

$$= \int \frac{e^{-3x}}{e^{-3x}-1} dx$$

$$= \int \frac{1}{u} \frac{du}{-3}$$

$$= \frac{-1}{3} \int \frac{du}{u}$$

$$= \frac{-1}{3} \ln|u| + C$$

$$= \frac{-1}{3} \ln|e^{-3x}-1| + C$$

$$u = e^{-3x} - 1$$

$$du = -3e^{-3x} dx$$

$$\frac{du}{-3} = e^{-3x} dx$$

Question 9. (5 marks) Determine the convergence or divergence of the sequence with the given n^{th} term. If the sequence converges find its limit.

$$b_n = \left(1 + \frac{1}{n}\right)^n \quad y = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

$$\ln y = \ln \left(\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \right)$$

$$\ln y = \lim_{n \rightarrow \infty} \ln \left(1 + \frac{1}{n}\right)^n$$

$$\ln y = \lim_{n \rightarrow \infty} n \ln \left(1 + \frac{1}{n}\right) \quad \text{has IF } \infty \cdot 0$$

$$\ln y = \lim_{n \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{n}\right)}{\frac{1}{n}} \quad \text{has IF } \frac{\infty}{\infty} \quad \text{so we use L'Hôpital's Rule}$$

$$\ln y = \lim_{n \rightarrow \infty} \frac{\frac{1}{1 + \frac{1}{n}} \left(\frac{-1}{n^2}\right)}{\left(\frac{-1}{n^2}\right)}$$

$$\ln y = 1$$

$$e^{\ln y} = e^1$$

$$y = e$$

Bonus Question. (3 marks)

$$\int \frac{\sqrt{1-x}}{\sqrt{x}} dx$$

$$\int \frac{\sqrt{1-x}}{\sqrt{x}} dx$$

$$\text{let } u = \sqrt{x}$$
$$du = \frac{dx}{2\sqrt{x}}$$

$$2\sqrt{x} du = dx$$

$$= \int \frac{\sqrt{1-(\sqrt{x})^2}}{\sqrt{x}} dx$$

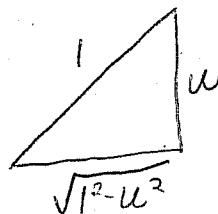
$$= \int \frac{\sqrt{1-u^2}}{\sqrt{x}} 2\sqrt{x} du$$

$$= 2 \int \sqrt{1-u^2} du$$

$$u = \sin \theta$$
$$du = \cos \theta d\theta$$

$$= 2 \int \sqrt{1-\sin^2 \theta} \cos \theta d\theta$$

$$= 2 \int \sqrt{\cos^2 \theta} \cos \theta d\theta$$



$$\therefore \cos \theta = \sqrt{1-u^2}$$

$$= 2 \int \cos^2 \theta d\theta$$

$$= 2 \int \frac{1 + \cos 2\theta}{2} d\theta$$

$$= \int 1 + \cos 2\theta d\theta$$

$$= \theta + \frac{\sin 2\theta}{2} + C$$

$$= \theta + \frac{2 \sin \theta \cos \theta}{2} + C$$

$$= \arcsin u + \sin \theta \cos \theta + C$$

$$= \arcsin \sqrt{x} + u \sqrt{1-u^2} + C$$

$$= \arcsin \sqrt{x} + \sqrt{x} \sqrt{1-x} + C$$