Student Number : ______________ Last Name only (print) : ____________________________

Other name(s) : ____________________________

FACULTY OF SCIENCE – FINAL EXAMINATION

MATHEMATICS 133 – VECTORS, MATRICES AND GEOMETRY

VERSION 1

Examiner: Professor I. Klemes
Associate Examiner: Professor C. Roth

Date: Monday, 5 December 2005.
Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

1. Check that this paper has all 14 questions, and a total of 9 pages which include this cover and the two continuation pages at the end.
2. This is a closed book exam. Notes, calculators, or any other devices, are not permitted.
3. Information on the computer card is to be entered with a soft lead pencil. Any erasing must be done cleanly. The computer will also accept black and blue ball point pens, but this is not advised as you will not be able to erase these to make corrections.
4. This exam paper is **Version 1**. Make sure that this same version number is filled in in the Version column of your computer card.
5. Print your student number and names in the blanks at the top of this exam paper, and on any rough work booklets provided.
6. Enter the requested ID information on the computer card, including the "check bits" columns as instructed on the card. Sign the computer card in the space indicated.
7. This paper may **not** be removed from the exam room by the student. All items, including rough work in the exam booklets, must be handed in.
8. This exam paper has a total of 100 points and consists of two parts:
   Part I consists of 10 multiple choice questions (1 to 10) worth 4 points each. **Only the answers entered on the computer card will be considered for Part I.**
   Part II consists of 4 written questions (11,12,13,14) worth 15 points each. In this part, show all of the work necessary for each step in the solution, and simplify answers.
9. Your solutions to the questions of Part II are to be written in the blank spaces provided in this exam paper below each question. *If you need to continue your work, use the facing page (i.e. back of the previous sheet)* Further work may be continued on the two blank continuation pages at the end of this exam paper. You must clearly indicate where your work will continue.
10. The examination Security Monitor Program detects pairs of students with unusually similar answer patterns on multiple choice exams. Data generated by this program can be used as admissible evidence, either to initiate or corroborate an investigation or a charge of cheating under Section 16 of the Code of Student Conduct and Disciplinary Procedures.

Please do not write inside the box :

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12	
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Part I: Answer on the computer card. Only the answer on the computer card will be considered. For each question the correct answer is assigned 4 points. Anything else (wrong answer, more than one answer, no answer, answer not readable by the machine, etc.) is assigned zero.

1. Given
$$A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B^{-1} = \begin{bmatrix} 4 & 1 \\ 3 & 4 \end{bmatrix},$$
compute the matrix $(AB)^{-1}$.

(A) $\begin{bmatrix} 1 & -5 \\ 2 & 3 \end{bmatrix}$, (B) $\begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ -5 & 2 \end{bmatrix}$, (C) $\begin{bmatrix} \frac{1}{2} & \frac{-5}{2} \\ 2 & 3 \end{bmatrix}$, (D) $\begin{bmatrix} 1 & 2 \\ \frac{-5}{2} & \frac{3}{2} \end{bmatrix}$, (E) $\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}$

2. A and B are 4×4 matrices with $\det A = 2$ and $\det B = -1$. Find
$$\det(3AB^TA^{-2}BA^TB^{-1}).$$

(A) 81, (B) -3, (C) 48, (D) -48, (E) -81

3. An equation of the plane containing the point $P(1,-1,2)$ and the line $x = 4$, $y = -1 + 2t$, $z = 2 + t$ is :

(A) $x + y - 2z + 5 = 0$, (B) $y - 2z + 5 = 0$, (C) $7x - 3y + z = 0$,

(D) $8x - 7y + z = 0$, (E) $7x + 8y + z = 0$

4. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$ such that $\|\mathbf{u}\| = 5$, $\|\mathbf{v}\| = 2$ and $\mathbf{u} \cdot \mathbf{v} = -8$. Then $\|\mathbf{u} \times \mathbf{v}\|$ is :

(A) 5, (B) 60, (C) 4, (D) 6, (E) 16

5. Find the distance from the point $(5,4,7)$ to the line containing the two points $(3,-1,2)$ and $(3,1,1)$.

(A) 7, (B) 9, (C) 11, (D) 5, (E) 3

Answer on the computer card. Exam continues overleaf...

6. Let A be the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ -1 & 2 & 1 & 2 & 3 \\ 2 & -4 & 0 & 2 & 1 \end{bmatrix}$$

Let r be the rank of A and q the dimension of the null space of A (i.e the nullity of A). What are the values of r and q ?

(A) $r = 2,\ q = 2$ (B) $r = 3,\ q = 2$ (C) $r = 2,\ q = 3$

(D) $r = 1,\ q = 4$ (E) $r = 3,\ q = 3$

7. For what values of α are the vectors $(\alpha, 1, 1)$, $(1, \alpha, \alpha)$ and $(8, 7, \alpha)$ linearly dependent ?

(A) $0, 1, 7$ (B) $-7, 1, -1$ (C) $7, -7, 1$ (D) $7, -7, -1$ (E) $7, 1, -1$

8. If we apply the Gram-Schmidt process to the vectors $(3, 0, 4, 0)$, $(-1, 0, 7, 0)$, $(6, 3, -5, 4)$ in that order, to obtain the orthonormal vectors $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$ in that order, then $\mathbf{u_2}$ is :

(A) $\frac{1}{5}(0, 3, 0, 4)$ (B) $\frac{1}{\sqrt{50}}(-1, 0, 7, 0)$ (C) $\frac{1}{5}(0, 3, 4, 0)$

(D) $\frac{1}{5}(-4, 0, 3, 0)$ (E) $(-4, 0, 3, 0)$

9. Which of the following equations has as its roots the eigenvalues of the matrix $\begin{bmatrix} 0 & 0 & 4 \\ 1 & 0 & -8 \\ 0 & 1 & 5 \end{bmatrix}$?

(i.e. Which one is the characteristic equation or its negative ?)

(A) $\lambda^3 + 8\lambda^2 - 4\lambda - 5 = 0$ (B) $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$ (C) $-5\lambda^3 + 8\lambda^2 - 4\lambda + 1 = 0$

(D) $-\lambda^3 - 8\lambda^2 + \lambda + 4 = 0$ (E) $\lambda^3 - \lambda^2 + \lambda - 1 = 0$

10. Given linearly independent vectors $\mathbf{v}$ and $\mathbf{w}$, which one of the statements below is always correct when

$$\mathbf{u} = \mathbf{v} - \left(\frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\right)\mathbf{w} \quad ?$$

(A) $\mathbf{u}$ is orthogonal to $\mathbf{v}$.

(B) $\mathbf{u}$ is the cross product of $\mathbf{v}$ and $\mathbf{w}$.

(C) $\mathbf{u}$ is orthogonal to $\mathbf{w}$.

(D) $\mathbf{u}$ is a unit vector.

(E) $\mathbf{w}$ is orthogonal to $\mathbf{v}$.

Answer on the computer card. Exam continues overleaf...

Part II: Answer in the space provided below each question. Continue on the facing page (i.e. back of the previous page). After that, use one of the continuation pages at the end of this booklet, being sure to indicate its location.

11. (15 points) Let P be the plane with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$.

    (a) Find a normal vector for the plane P.

    (b) Find the distance from the point $A = [5, -3, 5]^T$ to the plane P.

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then on page number: ______.

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12. (15 points) Consider the line L given by the equation $3x + 2y = 0$. Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal projection onto the line L, and let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection in the line L.

(a) Find the standard matrix of the projection P.

(b) Find two linearly independent eigenvectors of the reflection R and their corresponding eigenvalues.

(c) Find the standard matrix of the reflection R.

Reminder : For the standard matrices in (a) and (c), just writing down a formula for the answer from memory will not earn any partial marks if the formula is incorrect. As always, it is safer to show the work necessary to arrive at an answer.

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13. (15 points) Suppose that the vectors $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ are linearly independent.

(a) If $\mathbf{w_1} = \mathbf{v_1} - \mathbf{v_2} + \mathbf{v_3}$, $\mathbf{w_2} = \mathbf{v_1} - \mathbf{v_3}$, and $\mathbf{w_3} = \mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}$, show that the vectors $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ are linearly independent.

(b) If $\mathbf{a_1} = 4\mathbf{v_1} + 2\mathbf{v_2} - 2\mathbf{v_3}$, $\mathbf{a_2} = \mathbf{v_1} + 3\mathbf{v_2} - \mathbf{v_3}$, and $\mathbf{a_3} = 3\mathbf{v_1} - 6\mathbf{v_2}$, show that the vectors $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$ are not linearly independent.

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14. (15 points) Let $A = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$.

(a) Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$.

(b) Find m and M, the minimum and maximum values respectively of $q(X) = X^T AX$ on the unit circle $\|X\| = 1$.

(c) Give the coordinates of all the points X on the unit circle where the minima and maxima occur.

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then on page number: ______.

Exam continues overleaf...

Continuation of solution for problem: ______.
You must refer to this page on the page where the problem is printed.

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then on page number: ______.

Exam continues overleaf...

Continuation of solution for problem: ______.
You must refer to this page on the page where the problem is printed.

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then on the back of this page.

Last page of exam.