1. (a) Find the distance from the point $(2,0,1)$ to the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix}$.

   (b) Find an equation of the plane passing through the points $(1,0,-2)$, $(2,0,-3)$, and $(3,4,0)$. State your answer in the form $ax + by + cz = d$.

2. Let $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$, and consider the vectors $\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} -2 \\ 3 \\ -5 \end{bmatrix}$.

   (a) Find all possible ways of writing $\mathbf{v}$ as a linear combination of $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$. In other words, find all triples (a,b,c) that satisfy the equation $\mathbf{v} = a\mathbf{v_1} + b\mathbf{v_2} + c\mathbf{v_3}$.

   (b) Using your answer in (a), express $\mathbf{v}$ as a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$ only.

   (c) Using your answer in (a), express $\mathbf{v}$ as a linear combination of $\mathbf{v_1}$ and $\mathbf{v_3}$ only.

3. Let $A = \begin{bmatrix} 0 & -1 & -2 & 1 \\ -3 & 1 & 5 & -1 \\ 2 & 0 & -2 & 0 \\ 6 & -1 & -8 & 1 \end{bmatrix}$.

   (a) Find a basis for the row space of A.

   (b) Find a basis for the column space of A.

   (c) Find a basis for the null space of A.

4. Suppose that the set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a basis for a subspace V in $\mathbb{R}^4$.

   (a) If $\mathbf{w_1} = \mathbf{v_1}$, $\mathbf{w_2} = \mathbf{v_1} + \mathbf{v_2}$, and $\mathbf{w_3} = \mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}$, show that the set of vectors $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ is also a basis for V.

   (b) If $\mathbf{a_1} = \mathbf{v_1} + 2\mathbf{v_2}$, $\mathbf{a_2} = 3\mathbf{v_2} - \mathbf{v_3}$, and $\mathbf{a_3} = \mathbf{v_1} - \mathbf{v_2} + \mathbf{v_3}$, show that the set of vectors $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$ is **not** a basis for V.

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection in the line $2x + 3y = 0$.

   (a) Find the standard matrix A of the transformation T.

   (b) Use geometric reasoning to find the eigenvalues and eigenvectors of A.

   (c) Verify by direct computation that the answers in (b) are indeed correct.

more questions on next page ...

6. In each of the following, determine whether the given matrix is diagonalizable, and justify your answers:

$$\text{(a)}\quad A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \text{(b)}\quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

7. Let A be a 2 x 2 matrix such that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 3, and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue 2.

   (a) Find a matrix A satisfying all of the given conditions. State your final answer in the form $A = \begin{bmatrix} - & - \\ - & - \end{bmatrix}$.

   (b) Find a formula for A^n, valid for all positive integers n. State your final answer in the form $A^n = \begin{bmatrix} - & - \\ - & - \end{bmatrix}$.

   (c) Using the formula for A^n found in (b), compute $\det(A^n)$. Then compute $\det(A^n)$ using an alternate method. Do your results agree ?

8. Let $\mathbf{v} = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix}$, and $W = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

   (a) Find the orthogonal projection of $\mathbf{v}$ onto W.

   (b) Find the component of $\mathbf{v}$ orthogonal to W.

9. Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$.

   (a) Find the eigenvalues and eigenvectors of A. Hint: One of the eigenvalues has algebraic multiplicity 2.

   (b) Find an **orthogonal** matrix P and constants c_1, c_2, and c_3 such that the change of variables $X = PY$ changes the quadratic form $X^T AX$ into the form $c_1 y_1^2 + c_2 y_2^2 + c_3 y_3^2$. (Here $X = [x_1, x_2, x_3]^T$ and $Y = [y_1, y_2, y_3]^T$.)

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 133

Vectors, Matrices, and Geometry

Examiner: Prof. I. Klemes — Date: Monday, 6 December, 2004
Associate Examiner: Prof. D. Serbin — Time: 2pm - 5pm

INSTRUCTIONS

Show all necessary steps and details in your work.
Simplify your final answers as far as possible.
There are 9 questions, each worth 10 points.
Answer all questions in the examination booklets.
Calculators are not permitted.
This is a closed book examination.
Keep this exam paper when finished.

This exam comprises the cover and 2 pages of questions.