1. (a) Find the point P in the plane $x + y + z = -2$ which is closest to the point $Q = (2, 1, -3)$.
   (b) Find the distance from Q to the plane in part (a).

2. Find (giving justification) all values (if any) of the real parameter k such that the following system of equations for (x, y, z) has
   (a) a unique solution,
   (b) no solutions,
   (c) infinitely many solutions:

$$\begin{aligned} x + ky &= 1 \\ y + kz &= 2 \\ z + kx &= 3 \end{aligned}$$

3. Let A be the matrix $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$.
   (a) Write A^{-1} as a product of elementary matrices.
   (b) Write A as a product of elementary matrices.

4. Find the standard matrix of each of the following linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$.
   (a) Projection onto the line $4x_1 + x_2 = 0$.
   (b) Reflection in the line $4x_1 + x_2 = 0$.
   (c) Counterclockwise rotation by an angle of $\pi/3$ about the origin.

5. (a) Let $T : \mathbb{R}^7 \to \mathbb{R}^4$ be a linear transformation and let $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ be linearly **dependent** vectors in $\mathbb{R}^7$. Prove that $T(\mathbf{v_1}), T(\mathbf{v_2}), T(\mathbf{v_3})$ are linearly dependent vectors in $\mathbb{R}^4$.
   (b) Give an example of a nonzero linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ and an example of three linearly **independent** vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ in $\mathbb{R}^3$ such that the vectors $S(\mathbf{v_1})$, $S(\mathbf{v_2})$, $S(\mathbf{v_3})$ are linearly **dependent.**

6. (a) If A is a 15×14 matrix of rank 8, find the nullity of A.

   (b) If A and B are 3×3 matrices with $\det(A) = 2$ and $\det(B) = -5$, find $\det((A^T)^4(3B)^{-1})$.

   (c) If $-7, -5, -3$ are eigenvalues of the 4×4 matrix A with $\det(A) = 105$, find the fourth eigenvalue of A.

7. Let A be the matrix

$$A = \begin{bmatrix} -5 & -1 & 2 \\ 2 & -1 & -2 \\ -5 & -1 & 2 \end{bmatrix}.$$

   (a) Find the characteristic polynomial of A. (Hint: One eigenvalue of A is 0.)

   (b) Diagonalize A. (i.e. find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

8. (a) By applying the Gram-Schmidt process to the following set $\{\mathbf{v_1}, \mathbf{v_2}\}$ of vectors, find an orthonormal basis of their span.

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \ \mathbf{v_2} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

   (b) Complete the orthonormal set found in part (a) to an orthonormal basis of $\mathbb{R}^3$ using the Gram-Schmidt process. (Do not use the cross product.)

   (c) With respect to the orthonormal basis of $\mathbb{R}^3$ found in part (b), find the co-ordinates of the vector $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 133

Vectors, Matrices, and Geometry

Examiner: Prof. I. Klemes
Associate Examiner: Prof. W. Jonsson

Date: Thursday, 15 April, 2004
Time: 9:00 - 12:00

INSTRUCTIONS

There are 8 questions, each worth 10 points.
Answer all questions in the examination booklets.
Calculators are not permitted.
This is a closed book examination.
Keep this exam paper when finished.

This exam comprises the cover and 2 pages of questions.