LAST NAME: SOLUTIONS

FIRST NAME: ____________

STUDENT NUMBER: ____________

TEST 2 (A)

DAWSON COLLEGE

201-NYC-05 S2 - Linear Algebra

Instructor: E. Richer

Date: July 3rd 2008

Question 1. *(10 marks)*

Solve the matrix equation. (Note: X is a matrix, not a scalar)

$$X\begin{bmatrix} -1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 4 \\ & 9 & 10 \end{bmatrix}$$

X IS A 2X2 MATRIX $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -a+4b & 3a & 2a+b \\ -c+4d & 3c & 2c+d \end{bmatrix} = \begin{bmatrix} 7 & 3 & 4 \\ & 9 & 10 \end{bmatrix}$$

$3a = 3 \quad a = 1$

$3c = 9 \quad c = 3$

$-a + 4b = 7$

$-1 + 4b = 7$

$4b = 8$

$b = 2$

$-c + 4d = 13$

$-3 + 4d = 13$

$4d = 16$

$d = 4$

$$X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

CHECK

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 4 \\ 13 & 9 & 10 \end{bmatrix}$$

ZA2

Question 2. *(10 marks)*
The following system of linear equations has a unique solution. Find the solution using the inverse of the coefficient matrix. (Note: No other method will earn you any points)

$x+y+z=3$
$x+y-2z=3$
$-2x+y+z=-3$

$$Ax = b$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}$$

A^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 & 1 & 0 \\ -2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 - R_1 \to R_2 \\ R_3 + 2R_1 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -3 & -1 & 1 & 0 \\ 0 & 3 & 3 & 2 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} 3R_1 \to R_1 \\ R_2 \leftrightarrow R_3 \end{array} \left[\begin{array}{ccc|ccc} 3 & 3 & 3 & 3 & 0 & 0 \\ 0 & 3 & 3 & 2 & 0 & 1 \\ 0 & 0 & -3 & -1 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 + R_3 \to R_1 \\ R_2 + R_3 \to R_2 \end{array} \left[\begin{array}{ccc|ccc} 3 & 3 & 0 & 2 & 1 & 0 \\ 0 & 3 & 0 & 1 & 1 & 1 \\ 0 & 0 & -3 & -1 & 1 & 0 \end{array}\right]$$

$$R_1 - R_2 \to R_1 \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & -1 \\ 0 & 3 & 0 & 1 & 1 & 1 \\ 0 & 0 & -3 & -1 & 1 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & 0 \end{array}\right]$$

$$x = A^{-1} b = \begin{bmatrix} \frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ \\ 0 \end{bmatrix}$$

$$(x, y, z) = (2, \ , 0)$$

2A2

Question 3. *(10 marks)*

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

Find elementary matrices E_1, E_2 and E_3 that satisfy the following.

(i) $E_1 A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & 3 & 3 \end{bmatrix}$

(ii) $E_2 A = \begin{bmatrix} 3 & 3 & 3 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

(ii) $E_3 A = \begin{bmatrix} -4 & -4 & -4 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

(i) $R_2 - 3R_1 \to R_2$ $\qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

OR

$-\frac{1}{2} R_2 \to R_2$ $\qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(etc....)

(ii) $R_1 \leftrightarrow R_3$ $\qquad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(iii) $-4R_1 \to R_1$ $\qquad E_3 = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Question 4. *(10 marks)*

Let $A = \begin{bmatrix} 1 & 1 & -2 & 1 \\ 1 & 2 & 0 & \frac{1}{2} \\ -2 & 4 & 2 & 2 \\ -3 & 0 & -4 & -1 \end{bmatrix}$

Evaluate the **determinant** of A by **reducing the matrix to row echelon form**.

$R_2 - R_1 \to R_2$
$R_3 + 2R_1 \to R_3$
$R_4 + 3R_1 \to R_4$

$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 2 & -1/2 \\ 0 & 6 & -2 & 4 \\ 0 & 3 & -10 & 2 \end{bmatrix}$$

$R_3 - 6R_2 \to R_3$
$R_4 - 3R_2 \to R_4$

$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 2 & -1/2 \\ 0 & 0 & -14 & 7 \\ 0 & 0 & -16 & 7/2 \end{bmatrix}$$

$R_3 \cdot -1/14 \to R_3$

$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 2 & -1/2 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & -16 & 7/2 \end{bmatrix}$$

$R_4 + 16R_3 \to R_4$

$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 2 & -1/2 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & -9/2 \end{bmatrix} = B$$

$$\det B = -9/2$$

$$\det B = -1/14 \det A$$

$$\det A = -9/2(-14)$$

$$= 63$$

2A2

Question 5. *(10 marks)*
Let A and B be 4x4 matrices with $\det A = -4$ and $\det B = -2$.
Compute the following.
(i) $\det(2A^{-1}B^T)$
(ii) $\det((-2B)^{-1}(2A)^T)$

(i) $\det(2A^{-1}B^T)$

$= 2^4 \det A^{-1} \det B^T$

$= 2^4 \dfrac{1}{\det A} \det B = 2^4 \left(\dfrac{1}{-4}\right)(-2)$

$= 2^3 = \boxed{8}$

(ii) $\det((-2B)^{-1}(2A)^T)$

$= \det\left(-\frac{1}{2} B^{-1} 2 A^T\right)$

$= \det\left(-1\, B^{-1} A^T\right)$

$= (-1)^4 \dfrac{1}{\det B} \det A = \dfrac{1}{-2}(-4)$

$= \boxed{2}$

Question 6. *(10 marks)*
Given n x n matrices A and B. State whether or not the following statements are TRUE or FALSE. If the statement is false give an example otherwise simply state that the statement is true.

(a) $\det(A+B) = \det A + \det B$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A+B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\det A = 1$ $\det B = 1$ $\det A+B = 4$

$\det A + \det B = 2 \neq 4$

(b) $\det(AB) = \det(BA)$

TRUE

(c) If $A^2 = I$ then $A = I$ or $A = -I$

FALSE Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ $A^2 = I$

BUT $A \neq \pm I$

2A2

Question 7. *(10 marks)*
Let A be an nxn matrix. Prove that A is invertible **if and only if** AA^T is invertible.

See VERSION B or
CLASS NOTES

2A2

Question 8. *(10 marks)*

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ where $\det A = 4$. Find $\det B$ where $B = \begin{bmatrix} a-2g & b-2h & c-2i \\ 4d+2a & 4e+2b & 4f+2c \\ -2g & -2h & -2i \end{bmatrix}$.

$R_1 - R_3 \to R_1$ $\begin{bmatrix} a & b & c \\ 4d+2a & 4e+2b & 4f+2c \\ -2g & -2h & -2i \end{bmatrix}$

$-\frac{1}{2} R_3 \to R_3$ $\begin{bmatrix} a & b & c \\ 4d+2a & 4e+2b & 4f+2c \\ g & h & i \end{bmatrix}$

$R_2 - 2R_1 \to R_2$ $\begin{bmatrix} a & b & c \\ 4d & 4e & 4f \\ g & h & i \end{bmatrix}$

$1/4\, R_2 \to R_2$ $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = A$

$\det A = \left(-\frac{1}{2}\right)\left(\frac{1}{4}\right) \det B$

$\det B = \det A \,(-2)(4)$

$= 4(-2)(4)$

$= \boxed{-32}$