LAST NAME: SOLUTIONS

FIRST NAME: ____________

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TEST 2 (B)

DAWSON COLLEGE

201-NYC-05 S2 - Linear Algebra

Instructor: E. Richer

Date: July 3rd 2008

Question 1. *(10 marks)*

Solve the matrix equation. (Note: X is a matrix, not a scalar)

$$X\begin{bmatrix} -1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 6 & 6 & 6 \end{bmatrix}$$

2×3 $\quad$ 2×3 $\quad$ X must be a 2×2 matrix

Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -a+4b & 3a & 2a+b \\ -c+4d & 3c & 2c+d \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 6 & 6 & 6 \end{bmatrix}$$

By observation

$3a = 3 \quad 3c = 6$

$a = 1 \quad c = 2$

Then $-a+4b = 3$

$-1+4b = 3$

$4b = 4$

$b = 1$

& $-c+4d = 6$

$-2+4d = 6$

$4d = 8$

$d = 2$

So $X = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

CHECK: $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 6 & 6 & 6 \end{bmatrix}$

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Question 2. *(10 marks)*
The following system of linear equations has a unique solution. Find the solution using the inverse of the coefficient matrix. (Note: No other method will earn you any points)

$x+y+z=6$
$x+y-2z=9$
$-2x+y+z=-3$

$A\mathbf{x} = b$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 6 \\ 9 \\ -3 \end{bmatrix}$$

A^{-1} :

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 & 1 & 0 \\ -2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 - R_1 \to R_2 \\ R_3 + 2R_1 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -3 & -1 & 1 & 0 \\ 0 & 3 & 3 & 2 & 0 & 1 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 3 & 2 & 0 & 1 \\ 0 & 0 & -3 & -1 & 1 & 0 \end{array}\right] R_2 + R_3 \to R_2 \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 1 & 1 \\ 0 & 0 & -3 & -1 & 1 & 0 \end{array}\right]$$

$$3R_1 \to R_1 \left[\begin{array}{ccc|ccc} 3 & 3 & 3 & 3 & 0 & 0 \\ 0 & 3 & 0 & 1 & 1 & 1 \\ 0 & 0 & -3 & -1 & 1 & 0 \end{array}\right] R_1 + R_3 \to R_1 \left[\begin{array}{ccc|ccc} 3 & 3 & 0 & 2 & 1 & 0 \\ 0 & 3 & 0 & 1 & 1 & 1 \\ 0 & 0 & -3 & -1 & 1 & 0 \end{array}\right]$$

$$R_1 - R_2 \to R_1 \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & -1 \\ 0 & 3 & 0 & 1 & 1 & 1 \\ 0 & 0 & -3 & -1 & 1 & 0 \end{array}\right] \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & -1/3 \\ 0 & 1 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 1/3 & -1/3 & 0 \end{array}\right]$$

$$\mathbf{x} = A^{-1} b$$
$$= \begin{bmatrix} 1/3 & 0 & -1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & -1/3 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 9 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$$

So $(x, y, z) = (3, 4, -1)$

Question 3. *(10 marks)*

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

Find elementary matrices E_1, E_2 and E_3 that satisfy the following.

(i) $E_1A = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix}$

(ii) $E_2A = \begin{bmatrix} -2 & -2 & -2 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

(ii) $E_3A = \begin{bmatrix} 1 & 1 & 1 \\ 6 & 6 & 6 \\ 3 & 3 & 3 \end{bmatrix}$

(i) $R_1 \leftrightarrow R_2$ $\qquad E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(ii) $R_1 - R_3 \rightarrow R_1$ $\qquad E_2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(iii) $3R_2 \rightarrow R_2$ $\qquad E_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Question 4. *(10 marks)*

Let $A = \begin{bmatrix} 1 & 1 & -2 & 1 \\ 1 & 2 & 0 & \frac{1}{2} \\ -1 & 2 & 1 & 1 \\ -3 & 0 & -4 & -1 \end{bmatrix}$

Evaluate the **determinant** of A by **reducing the matrix to row echelon form**.

$R_2 - R_1 \to R_2$
$R_3 + R_1 \to R_3$
$R_4 + 3R_1 \to R_4$

$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 2 & -\frac{1}{2} \\ 0 & 3 & -1 & 2 \\ 0 & 3 & -10 & 2 \end{bmatrix}$$

$R_3 - 3R_2 \to R_3$
$R_4 - 3R_2 \to R_4$

$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 2 & -\frac{1}{2} \\ 0 & 0 & -7 & \frac{7}{2} \\ 0 & 0 & -16 & \frac{7}{2} \end{bmatrix} \quad -\tfrac{1}{7}R_3 \to R_3 \quad \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 2 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & -16 & \frac{7}{2} \end{bmatrix}$$

$R_4 + 16R_3 \to R_4$

$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 2 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{9}{2} \end{bmatrix} = B$$

$\det B = -\frac{9}{2}$

$\det B = -\frac{1}{7}\det A$

$\det A = -7 \det B = -7\left(-\frac{9}{2}\right)$
$= \frac{63}{2}$

$\boxed{\det A = \frac{63}{2}}$

Question 5. *(10 marks)*
Let A and B be 3x3 matrices with $\det A = 2$ and $\det B = -2$.
Compute the following.
(i) $\det(2A_{-1}B^T)$
(ii) $\det((-2B)^{-1}(2A)^T)$

(i) $\det(2A^{-1}B^T)$

$= 2^3 \det A^{-1} \det B^T$

$= 2^3 \frac{1}{\det A} \det B$

$= 2^3 \left(\frac{1}{2}\right)(-2) = -2^3 = -8$

(ii) $\det((-2B)^{-1}(2A)^T)$

$= \det\left(-\frac{1}{2}B^{-1} 2 A^T\right)$

$= \det\left(-1 B^{-1} A^T\right)$

$= (-1)^3 \frac{1}{\det B} \det A$

$= -1 \left(\frac{1}{-2}\right)(2) = 1$

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Question 6. *(10 marks)*
Given n x n matrices A and B. State whether or not the following statements are TRUE or FALSE. If the statement is false give an example otherwise simply state that the statement is true.
(a) $\det(AB) = \det(BA)$

TRUE

(b) $\det(A+B) = \det A + \det B$ FALSE

$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ $A+B = \begin{bmatrix} 2 & 2 \\ -2 & 0 \end{bmatrix}$

$\det A = 1$ $\det B = 1$ $\det(A+B) = 4$

$\det A + \det B = 2 \neq \det(A+B) = 4$

(c) If $A^2 = I$ then $A = I$ or $A = -I$

FALSE

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ $A^2 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

& $A \neq \pm I$

Question 7. *(10 marks)*
Let A be an nxn matrix. Prove that A is invertible **if and only if** AA^T is invertible.

($\Longrightarrow$) Suppose A is invertible,
then $\det A \neq 0$

Now consider AA^T
$\det(AA^T) = \det A \det A^T$
$= (\det A)^2 \neq 0$ because $\det A \neq 0$

So $\det AA^T \neq 0$ which means
AA^T is invertible

($\Longleftarrow$) Suppose AA^T is invertible,
Then $\det(AA^T) \neq 0$
So $(\det A)^2 \neq 0$ (because $\det AA^T = (\det A)^2$)
So $\det A \neq 0$
Thus A is invertible ∎

Question 8. *(10 marks)*

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ where $\det A = 4$. Find $\det B$ where $B = \begin{bmatrix} a+g & b+h & c+i \\ 3d+2a & 3e+2b & 3f+2c \\ -2g & -2h & -2i \end{bmatrix}$.

B: $-\frac{1}{2}R_3 \to R_3$ $\begin{bmatrix} a+g & b+h & c+i \\ 3d+2a & 3e+2b & 3f+2c \\ g & h & i \end{bmatrix}$

$R_1 - R_3 \to R_1$ $\begin{bmatrix} a & b & c \\ 3d+2a & 3e+2b & 3f+2c \\ g & h & i \end{bmatrix}$

$R_2 - 2R_1 \to R_2$ $\begin{bmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{bmatrix}$

$\frac{1}{3}R_2 \to R_2$ $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = A$

$\det A = \det B \left(-\frac{1}{2}\right)\left(\frac{1}{3}\right)$

$\det B = -2(3)\det A$

$= -6(4) = \boxed{-24}$