7B

LAST NAME: SOLUTIONS

FIRST NAME: ______________

STUDENT NUMBER: ______________

TEST 3 (B)

DAWSON COLLEGE

201-NYC-05 Linear Algebra

Instructor: E. Richer

Date: July 17th 2008

Question 1. *(10 marks)*

Let $\vec{u} = (1,2,-4)$ and $\vec{v} = (2,-1,-1)$.

(a) Find the cosine of the angle between u and v.

(b) Find the area of the triangle defined by $\vec{u}$ and $\vec{v}$.

(a)

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$$

$$= \frac{2-2+4}{\sqrt{1^2+2^2+(-4)^2}\sqrt{2^2+(-1)^2+(-1)^2}}$$

$$= \boxed{\frac{4}{\sqrt{21}\sqrt{6}}}$$

(b)

$$\text{Area} = \frac{1}{2}\|\vec{u}\times\vec{v}\|$$

$$= \frac{1}{2}\left\| \begin{pmatrix}1\\2\\-4\end{pmatrix} \times \begin{pmatrix}2\\-1\\-1\end{pmatrix} \right\|$$

$$= \frac{1}{2}\|(-6,-7,-5)\|$$

$$= \frac{1}{2}\sqrt{(-6)^2+(-7)^2+(-5)^2} = \boxed{\frac{1}{2}\sqrt{110}}$$

Question 2. *(10 marks)*
Find the equation of the plane containing the points $A(1,2,-1)$ containing the line $(x,y,z) = (1+t, 2-t, 0)$

$B = (1,2,0)$ is on the line

$\vec{AB} = (0,0,1)$

$\vec{d} = (1,-1,0)$ direction vector of line

$\vec{n} = \vec{AB} \times \vec{d} = (1,1,0)$

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$x + y + d = 0$

$A(1,2,-1)$ $\quad 1 + 2 + d = 0$

$3 + d = 0$

$d = -3$

$$\boxed{x + y - 3 = 0}$$

Question 3. *(10 marks)*
Find the intersection of the planes $2x - y + 3z = 0$ and $x - y - z + 2 = 0$ and give a geometric interpretation of this intersection.

$$2x - y + 3z = 0$$
$$x - y - z = -2$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 1 & -1 & -1 & -2 \end{array}\right] \quad R_2 - \tfrac{1}{2}R_1 \to R_2$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 0 & -\tfrac{1}{2} & -\tfrac{5}{2} & -2 \end{array}\right] \quad \begin{array}{l} \tfrac{1}{2}R_1 \to R_1 \\ -2R_2 \to R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\tfrac{1}{2} & \tfrac{3}{2} & 0 \\ 0 & 1 & 5 & 4 \end{array}\right]$$

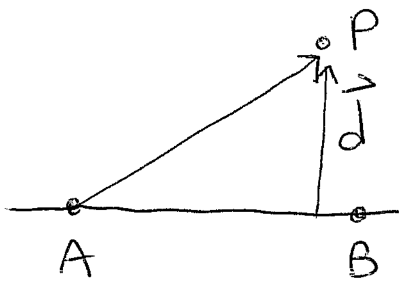
Let $z = t$

$$y = 4 - 5t$$
$$\begin{aligned} x &= \tfrac{1}{2}y - \tfrac{3}{2}z \\ &= \tfrac{1}{2}(4-5t) - \tfrac{3}{2}t \\ &= 2 - \tfrac{5}{2}t - \tfrac{3}{2}t \\ &= 2 - \tfrac{8}{2}t = 2 - 4t \end{aligned}$$

$$(x, y, z) = (2-4t, 4-5t, t)$$

the intersection is A line.

Question 4. *(10 marks)*
Find the distance between the point $P(1,1,2)$ and the line passing through the points $A(1,2,3)$ and $B(-1,-1,0)$.



$$\vec{AP} = (0,-1,-1)$$
$$\vec{AB} = (-2,-3,-3)$$

$$\text{proj}_{\vec{AB}} \vec{AP} = \frac{\vec{AP}\cdot\vec{AB}}{\vec{AB}\cdot\vec{AB}} \vec{AB}$$
$$= \frac{(0+3+3)}{(4+9+9)}(-2,-3,-3)$$
$$= \left(\frac{6}{22}\right)(-2,-3,-3)$$
$$= \frac{3}{11}(-2,-3,-3)$$

$$\vec{d} = \vec{AP} - \text{proj}_{\vec{AB}} \vec{AP}$$
$$= (0,-1,-1) - \frac{3}{11}(-2,-3,-3)$$
$$= \left(\frac{6}{11}, \frac{-2}{11}, \frac{-2}{11}\right) = \frac{2}{11}(3,-1,-1)$$

$$\|\vec{d}\| = \frac{2}{11}\sqrt{3^2+(-1)^2+(-1)^2} = \boxed{\frac{2}{11}\sqrt{11}}$$

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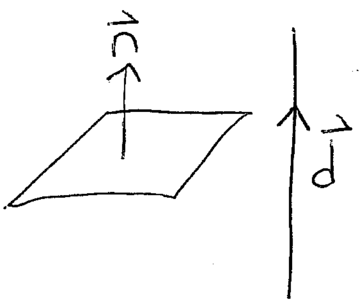
Question 5. *(10 marks)*
Determine whether the plane $2x - y + 3z + 3 = 0$ is perpendicular to the line $(x, y, z) = (2 - 4t, 2t, 1 - 6t)$. Explain your answer.

the NORMAL OF THE plANE
is $\vec{n} = (2, -1, 3)$

THE direction VECTOR OF THE line is

$\vec{d} = (-4, 2, -6)$

$\vec{n}$ & $\vec{d}$ are parallel



so the line & plANE
$\boxed{\text{ARE perpendicular}}$

Question 6. *(10 marks)*
Find parametric equations for the line passing through the point $P(1,2,2)$ that is parallel to the planes $2x-y+z+1=0$ and $x+y+z+2=0$

the NORMALS OF THE plane Are

$\vec{n_1} = (2,-1,1)$
$\vec{n_2} = (1,1,1)$

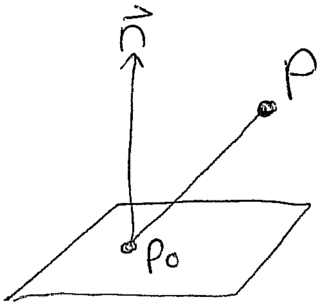
direction vector $\vec{d}$ OF THE line is

$$\vec{d} = \vec{n_1} \times \vec{n_2} = (-2,-1,3)$$
$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

THE EQUATIONS are

$$\boxed{(x,y,z) = (1,2,2) + (-2,-1,3)t}$$

Question 7. *(10 marks)*
Find the distance between the plane $2x-3y+4z=5$ and the point $P(1,-1,2)$.



Pick P_0 on the plane

$P_0(1,-1,0)$

$\overrightarrow{P_0P} = (0,0,2)$ $\quad \vec{n} = (2,-3,4)$

$$\text{Proj}_{\vec{n}} \overrightarrow{P_0P} = \frac{\vec{n}\cdot\overrightarrow{P_0P}}{\vec{n}\cdot\vec{n}}\,\vec{n}$$

$$= \frac{8}{29}(2,-3,4)$$

$$\text{distance} = \left\| \frac{8}{29}(2,-3,4) \right\|$$

$$= \frac{8}{29}\sqrt{2^2+(-3)^2+4^2}$$

$$\boxed{= \frac{8\sqrt{29}}{29}}$$

Question 8.

(a)*(5 marks)*

Let V be the set of all pairs of real numbers (x,y) with the operations:

$(x,y)+(x',y')=(xx',yy')$

$k(x,y)=(kx,ky)$

What is the zero object O_v of this vector space?

(b) *(5 marks)*

Let V be the set of all pairs of real numbers (x,y) with operations defined as follows:

$(x,y)+(x',y')=(x+x',y+y')$

$k(x,y)=(0,ky)$

Prove that V is NOT a vector space.

(c) Let V be the set of all 2x2 matrices of the form $\begin{bmatrix} a & b \\ a+b & 0 \end{bmatrix}$. Prove that V satisfies axiom 1 of vector spaces. (See the back page for vector space axioms)

(a) $O_V = (1,1)$

because $(x,y)+(1,1) = (x,y)$

(b) AXIOM 10 FAILS

Say $(x,y) = (1,1)$

$1(x,y) = 1(1,1)$

$= (0,1) \neq (x,y)$

(c) Let $\begin{bmatrix} a & b \\ a+b & 0 \end{bmatrix}$ & $\begin{bmatrix} c & d \\ c+d & 0 \end{bmatrix}$ be in V

then $\begin{bmatrix} a & b \\ a+b & 0 \end{bmatrix} + \begin{bmatrix} c & d \\ c+d & 0 \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+b+c+d & 0 \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ (a+c)+(b+d) & 0 \end{bmatrix}$

IB

BONUS *(5 marks)*
Prove that $\vec{u}\cdot(\vec{v}\times\vec{w}) = -(\vec{u}\times\vec{z})\cdot\vec{v}$.

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \det\begin{bmatrix}\vec{u}\\ \vec{v}\\ \vec{w}\end{bmatrix}$$

$$= -\det\begin{bmatrix}\vec{v}\\ \vec{u}\\ w\end{bmatrix}$$

$$= -\vec{v}\cdot(\vec{u}\times\vec{w})$$

$$= -(\vec{u}\times\vec{w})\cdot\vec{v}$$

Vector Space Axioms

1- If u and v are objects in V, then $u+v$ is in V.

2- $u+v=v+u$

3- $u+(v+w)=(u+v)+w$

4- There is an object 0_v called a zero object for V such that $0_v+u=u$ for all u in V.

5- For each u in V, there is an object $-u$ in V called a negative of u such that $u+(-u)=0_v$

6- If k is any scalar and u is any object in V, then ku is in V

7- k(u+v) = ku + kv

8- (k+m)u = ku + mu

9- k(mu) = (km)u

10- 1u = u