LAST NAME: SOLUTIONS

FIRST NAME: _____

STUDENT NUMBER: _____

TEST 3 (B)
DAWSON COLLEGE
201-NYC-05 Linear Algebra
Instructor: E. Richer
Date: July 17th 2008

Question 1. *(10 marks)*
Let $\vec{u} = (1, 2, -4)$ and $\vec{v} = (2, -1, -1)$.
(a) Find the cosine of the angle between u and v.
(b) Find the area of the triangle defined by $\vec{u}$ and $\vec{v}$.

(a)
$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

$$= \frac{2 - 2 + 4}{\sqrt{1^2 + 2^2 + (-4)^2}\sqrt{2^2 + (-1)^2 + (-1)^2}}$$

$$= \boxed{\frac{4}{\sqrt{21}\sqrt{6}}}$$

(b)
$$\text{Area} = \frac{1}{2}\|\vec{u} \times \vec{v}\|$$

$$= \frac{1}{2}\left\| \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right\|$$

$$= \frac{1}{2}\|(-6, -7, -5)\|$$

$$= \frac{1}{2}\sqrt{(-6)^2 + (-7)^2 + (-5)^2} = \boxed{\frac{1}{2}\sqrt{110}}$$

Question 2. *(10 marks)*
Find the equation of the plane containing the points $A(1,2,-1)$
containing the line $(x,y,z) = (1+t, 2-t, 0)$

$B = (1,2,0)$ is on the line

$\vec{AB} = (0,0,1)$

$\vec{d} = (1,-1,0)$ direction vector of line

$\vec{n} = \vec{AB} \times \vec{d} = (1,1,0)$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$x + y \qquad + d = 0$

$A(1,2,-1) \qquad 1 + 2 \qquad + d = 0$

$3 + d = 0$

$d = -3$

$\boxed{x + y \qquad - 3 = 0}$

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Question 3. *(10 marks)*
Find the intersection of the planes $2x - y + 3z = 0$ and $x - y - z + 2 = 0$ and give a geometric interpretation of this intersection.

$$2x - y + 3z = 0$$
$$x - y - z = -2$$

$$\begin{bmatrix} 2 & -1 & 3 & | & 0 \\ 1 & -1 & -1 & | & -2 \end{bmatrix} \quad R_2 - \tfrac{1}{2}R_1 \to R_2$$

$$\begin{bmatrix} 2 & -1 & 3 & | & 0 \\ 0 & -\tfrac{1}{2} & -\tfrac{5}{2} & | & -2 \end{bmatrix} \quad \begin{array}{l} \tfrac{1}{2}R_1 \to R_1 \\ -2R_2 \to R_2 \end{array}$$

$$\begin{bmatrix} 1 & -\tfrac{1}{2} & \tfrac{3}{2} & | & 0 \\ 0 & 1 & 5 & | & 4 \end{bmatrix}$$

Let $z = t$
$$y = 4 - 5t$$
$$x = \tfrac{1}{2}y - \tfrac{3}{2}z$$
$$= \tfrac{1}{2}(4 - 5t) - \tfrac{3}{2}t$$
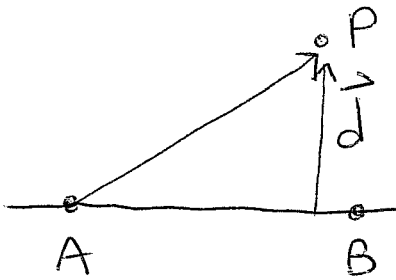$$= 2 - \tfrac{5}{2}t - \tfrac{3}{2}t$$
$$= 2 - \tfrac{8}{2}t = 2 - 4t$$

$$(x, y, z) = (2 - 4t, \ 4 - 5t, \ t)$$

the intersection is A line.

Question 4. *(10 marks)*

Find the distance between the point $P(1,1,2)$ and the line passing through the points $A(1,2,3)$ and $B(-1,-1,0)$.

$$\overrightarrow{AP} = (0,-1,-1)$$

$$\overrightarrow{AB} = (-2,-3,-3)$$

$$\text{proj}_{\overrightarrow{AB}}\, \overrightarrow{AP} = \frac{\overrightarrow{AP} \cdot \overrightarrow{AB}}{\overrightarrow{AB} \cdot \overrightarrow{AB}}\, \overrightarrow{AB}$$

$$= \frac{(0+3+3)}{(4+9+9)}\,(-2,-3,-3)$$

$$= \left(\frac{6}{22}\right)(-2,-3,-3)$$

$$= \frac{3}{11}(-2,-3,-3)$$

$$\vec{d} = \overrightarrow{AP} - \text{proj}_{\overrightarrow{AB}}\, \overrightarrow{AP}$$

$$= (0,-1,-1) - \frac{3}{11}(-2,-3,-3)$$

$$= \left(\frac{6}{11},\ -\frac{2}{11},\ \frac{-2}{11}\right) = \frac{2}{11}(3,-1,-1)$$

$$\|\vec{d}\| = \frac{2}{11}\sqrt{3^2+(-1)^2+(-1)^2} = \boxed{\frac{2}{11}\sqrt{11}}$$
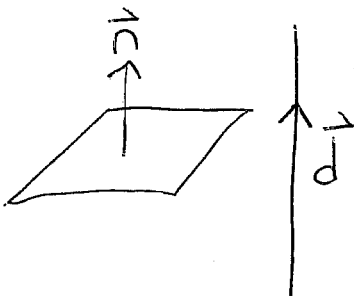
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Question 5. *(10 marks)*
Determine whether the plane $2x - y + 3z + 3 = 0$ is perpendicular to the line $(x, y, z) = (2 - 4t, 2t, 1 - 6t)$. Explain your answer.

the NORMAL OF THe plane

is $\vec{n} = (2, -1, 3)$

The direction vecTOR of THe line is

$$\vec{d} = (-4, 2, -6)$$

$\vec{n}$ & $\vec{d}$ are parallel

so the line & plane

ARE perpendicular

Question 6. *(10 marks)*

Find parametric equations for the line passing through the point $P(1,2,2)$ that is parallel to the planes $2x - y + z + 1 = 0$ and $x + y + z + 2 = 0$

The NOrMALS OF THE plane Are

$$\vec{n_1} = (2, -1, 1)$$
$$\vec{n_2} = (1, 1, 1)$$
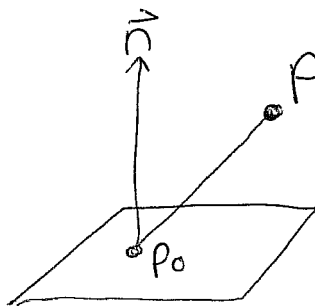
direction vector $\vec{d}$ OF THE line is

$$\vec{d} = \vec{n_1} \times \vec{n_2} = (-2, -1, 3)$$

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

THE EquATIONS are

$$(x, y, z) = (1, 2, 2) + (-2, -1, 3)t$$

Question 7. *(10 marks)*
Find the distance between the plane $2x - 3y + 4z = 5$ and the point $P(1, -1, 2)$.

Pick P_0 on the plane

$P_0 (1, -1, 0)$

$\overrightarrow{P_0P} = (0, 0, 2)$ $\vec{n} = (2, -3, 4)$

$\text{proj}_{\vec{n}} \overrightarrow{P_0P} = \dfrac{\vec{n} \cdot \overrightarrow{P_0P}}{\vec{n} \cdot \vec{n}} \vec{n}$

$= \dfrac{8}{29} (2, -3, 4)$

$\text{distance} = \left\| \dfrac{8}{29} (2, -3, 4) \right\|$

$= \dfrac{8}{29} \sqrt{2^2 + (-3)^2 + 4^2}$

$\boxed{= \dfrac{8}{29} \sqrt{29}}$

Question 8.

(a)*(5 marks)*

Let V be the set of all pairs of real numbers (x,y) with the operations:

$(x,y)+(x',y')=(xx',yy')$

$k(x,y)=(kx,ky)$

What is the zero object O_v of this vector space?

(b) *(5 marks)*

Let V be the set of all pairs of real numbers (x,y) with operations defined as follows:

$(x,y)+(x',y')=(x+x',y+y')$

$k(x,y)=(0,ky)$

Prove that V is NOT a vector space.

(c) Let V be the set of all 2x2 matrices of the form $\begin{bmatrix} a & b \\ a+b & 0 \end{bmatrix}$. Prove that V satisfies axiom 1 of vector spaces. (See the back page for vector space axioms)

(a) $O_v = (1,1)$

because $(x,y)+(1,1) = (x,y)$

(b) AXIOM 10 FAILS

Say $(x,y) = (1,1)$

$1(x,y) = 1(1,1)$

$\qquad = (0,1) \neq (x,y)$

(c) Let $\begin{bmatrix} a & b \\ a+b & 0 \end{bmatrix}$ & $\begin{bmatrix} c & d \\ c+d & 0 \end{bmatrix}$ be in V

then $\begin{bmatrix} a & b \\ a+b & 0 \end{bmatrix} + \begin{bmatrix} c & d \\ c+d & 0 \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+b+c+d & 0 \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ (a+c)+ & 0 \\ (b+d) & \end{bmatrix}$

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BONUS *(5 marks)*
Prove that $\vec{u} \cdot (\vec{v} \times \vec{w}) = -(\vec{u} \times \vec{z}) \cdot \vec{v}$.

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \det \begin{bmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{bmatrix}$$

$$= -\det \begin{bmatrix} \vec{v} \\ \vec{u} \\ \vec{w} \end{bmatrix}$$

$$= -\vec{v} \cdot (\vec{u} \times \vec{w})$$

$$= -(\vec{u} \times \vec{w}) \cdot \vec{v}$$

Vector Space Axioms

1- If u and v are objects in V, then $u+v$ is in V.

2- $u+v = v+u$

3- $u+(v+w) = (u+v)+w$

4- There is an object 0_v called a zero object for V such that $0_v + u = u$ for all u in V.

5- For each u in V, there is an object $-u$ in V called a negative of u such that $u+(-u) = 0_v$

6- If k is any scalar and u is any object in V, then ku is in V

7- $k(u+v) = ku + kv$

8- $(k+m)u = ku + mu$

9- $k(mu) = (km)u$

10- $1u = u$